“MATHEMATICS for EVERYONE” (Summary)

Author: Laurie Buxton (1984).

Trying to make math cool is like trying to make "not holding your breath for five minutes" cool. The goal of this summary is not to make math ‘cool’, but to present the fundamentals in a commonsense way with real world examples and applications (with a dash of ‘cool’ graphics).

I – STARTING OUT

CH1 – Back to Basics

Curiosities

If we add up a string of odd numbers, starting with 1, we always get a square (the square of an integer).
1 + 3 + 5 + 7 + 9 + 11 + 13 = 49

Primes

A prime number is a natural number greater than 1 that has no positive divisors other than 1 and itself. Primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97…

Fundamental Theorem of Arithmetic

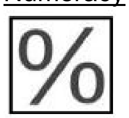
Any integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers.
2 x 2 = 4 | 2 x 3 = 6 | 2 x 2 x 2 = 8 | 3 x 3 = 9 | 2 x 5 = 10 | 2 x 2 x 3 = 12 | 2 x 7 = 14 ….

A very good approximation for the number of primes up to any number x is shown by the Prime Number Theorem (from 2 to x). Works best for large numbers. ($\ln(x) = \log_e(x)$).

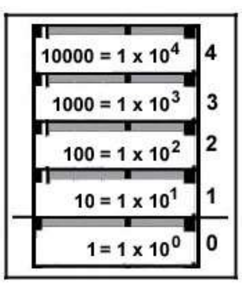
$$\int \frac{dx}{\ln(x)} = \ln(\ln(x)) + \ln(x) + \frac{(\ln(x))^2}{2 \cdot 2!} + \frac{(\ln(x))^3}{3 \cdot 3!} + \frac{(\ln(x))^4}{4 \cdot 4!} + \cdots$$

Not to shock the reader right off the start with visually intimating math, but there is something weird about an expression from the calculus being linked with primes (we’ll get to infinite series and calculus later).

Numeracy



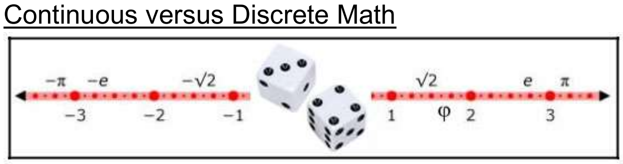
Percentages are often a source of public misunderstandings and a means for those who wish to persuade the public of things for their own commercial or political things.
% = baseline of 100.



Scientists write numbers as if they were all ‘ground floor’ numbers between 1 & 10 and then indicate which floor we need to take the lift to. 93,000,000 = 9.3 x 10^7. “Floors” = orders of magnitude.

Another important skill: to *estimate* rather than calculate.

Continuous versus Discrete Math



Continuous mathematics is math based on real numbers that have the property of varying "smoothly". In continuous math, you can always find a number between two other numbers; in fact, you can always find an infinite set of numbers between them. In discrete mathematics, objects, like integers, graphs or statements in logic, have distinct values. Discrete mathematics excludes topics in "continuous mathematics" such as calculus and analysis.

CH2 – SOME PRACTICALITIES

Money Matters

Issue: how much trouble are you prepared to go to get a good deal? Time is money.
Car buying: depreciation and loss on capital may matter more than the cost of gasoline.

Many people believe that a numerical understanding removes freedom of choice. This is nonsense. Numbers will tell you what things cost, not what you should do. “Who shall be master”? as Humpty Dumpty remarked.

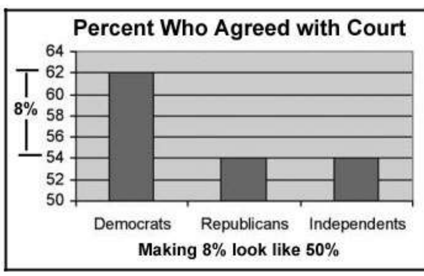
The Greeks made a distinction between ‘arithmetica’ – the study of whole numbers and their relationships – and ‘logistica’ – the calculation used by tradesmen and the lower class.

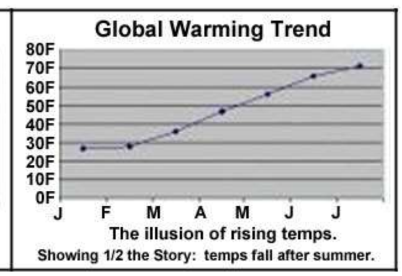
Health

There is nothing personal about statistics, but they have their role. Doctors plus computer-based help using statistical trends do better than doctors without. A real triumph of statistics lay in the establishment of the link between smoking and lung cancer. Issue: statistics do not offer proof, statistics offer probability.

Graphs and the Media

If a good proportion of the population does not comprehend, attempts to explain will fail. TV, radio and the press have points of view to express (and sales expectations) and will tailor the information to support their views.





Graph tricks: 1) how steep the line appears depends simply on the scale used on the x-axis, 2) difference in bar heights based on the scale used on the y-axis, 3) incomplete picture.
How to mislead: what scales you use and where you put the zeros.

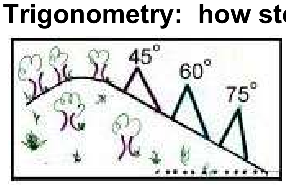
1. Do a quick math estimate and apply common sense
2. Check the authority of the source.
3. Question if the statistics are biased or statistically insignificant.
4. Question if the statistics are skewed purposely or misinterpreted.
5. Fully utilize your resources to conduct more research.

The left side of the brain is normally dominate and controls language, numerical work and most analytic thinking; the right gives us our spatial perceptions and may be a larger factor in creativity and problem-solving.

Sports

For cricket or baseball, performances are expressed in ‘averages’ calculated to at least two decimal places (the ‘dammed dots’).
Chess: Elo rating for grand masters and ordinary players.

Trigonometry: how steep is a hill?



Trigonometry is about ratios in triangles (‘triangle’ measurement). Once we attach a number to a notion, such as steepness, we have a measure.

CH3 – THE STORY OF CALCULATION

Numerals

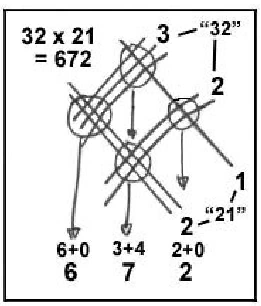
Babylonian's system of numeration: counting in sixties (60) remain in our measurement of time – 60 seconds per minute, 60 minutes per hour. Also, to divide a circle into 360 degrees. They used cuneiform (wedge-shape) symbols to represent numbers.

The Roman numeral system is basically a tens (10) system, but the symbols were not useful for basic arithmetic and definitely not useful for precision engineering.

The Hindi-Arabic system added the most important improvement: the symbol for zero.

In the medieval days, to be a 'scholar' meant to be able to read, write and figure. Early Liberal Arts studies began with 'Tridium' instruction; later on 'Quadrium' instruction was added. Note the dominate theme of numbers.

Tridium Instruction	Quadrium Instruction
Grammar	Arithmetic (numbers)
Logic	Geometry (numbers in space)
Rhetoric	Music (numbers in time)
-	Astronomy (numbers in space-time)



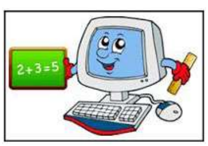
Napier was a 17th century Scottish mathematician who devised a system of rods to facility multiplication.

Logarithms

Logarithms are an obsolete method of computation: multiplication was reduced to addition. The exponents were added and the total was located in a table to find the answer. The affordable electronic calculator, PC & smartphone killed the logarithm.

$83 \times 236 = 10^{1.9191} \times 10^{2.3729} = 10^{(1.9191\,+\,2.3729)} = 10^{4.1020} = 19{,}588$

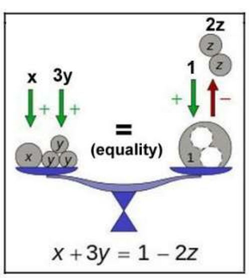
Math and the Computer



Tedious, repetitious work can be done with great rapidity, to any desired degree of accuracy, with a calculator or computer. By good guesswork and getting a result somewhere near the answer, we can derive a technique for getting a closer result. The process is repeated to get closer results.

Computers are very useful in solving complex equations where it is impossible or very difficult to solve by explicit algebraic formulas (eg, equations with powers of 5 or greater). A classic programming workhorse is iteration (a repeated routine). Another important use for the computer is infinite series calculations: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$.. (infinite series will be discussed further in Chapter 7).

Formula vs. Equation



A formula is a like a tool. It links one quantity to one or more other quantities. An example of formula is the area of circle: $A = \pi \times r^2$

An equation is like a puzzle. It is an equality containing one or more variables. Solving an equation consists of determining which values of the variables make the equality true. An analogy is a balance scale as shown in the left-side diagram – the goal of the equation is to 'balance' out the puzzle.

In algebra there are two main families of equations:

1) Linear ($y = mx + b$).
2) Polynomial (quadratic: $y = ax^2 + bx + c$)

II – THE CENTRAL CORE
CH5 – NUMBER

A clear understanding of the definitions and the nature of numbers is the magic key to understanding mathematics.

The Great Math Mystery

"How can it be that mathematics, a creation of the human mind independent of existence, should be so adapted to the objects of reality?" – Albert Einstein

Taking a philosophical turn on speculating on how numbers came to be, two camps are presented:
1) 'Math is Discovered', 2) 'Math is Invented':

1) Math Discovered. Natural numbers or the counting numbers basically started with the number 2. Within humans there is a strong symmetry in nature – two eyes, two hands, two legs, etc. So, the number 2 was more discovered than invented. This is the Platonic school of thought, dating back to Plato, where number is considered to be fundamental objective knowledge. It is the notion that numbers (and mathematical forms) underpin the physical universe and are out there waiting to be discovered (rather than invented). For mathematicians, in general, math is discovered.

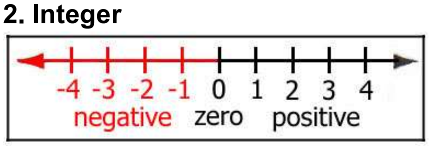
2) Math Invented. The idea is that number, beyond natural numbers, is an invented concept. Numbers (and mathematical forms) are objects of our human imagination and we make them up as we go along (eg, complex numbers), tailoring them to describe reality. Welcome to the world of abstraction. Philosophers label this perspective the 'Non-Platonists' view. In general, for engineers, and for physicists, sometimes reluctantly, math is invented.

So, is math a discovered part of the universe or a very human invention? As we'll see in the following chapters, people learned that the natural numbers have all kinds of *intricate* relationships – those were the discoveries. Humans invented the concept, but later discovered the relations among the different concepts. In the end, taking the rational approach, number is both – discovery and invention.

1. Natural Number

Natural numbers are the positive counting numbers: 1, 2, 3, 4… If we add zero, then they're called the whole natural numbers.

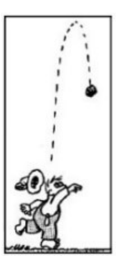
2. Integer



With integers we have to look at the 'other side' of the number line – the negative numbers. Integers are positive *and* negative whole natural numbers.

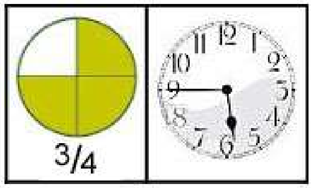
Negative Number

The Greek mathematician Diophantus (3rd-century BC) came across -4 as the solution of an equation and rejected it as absurd. This attitude persisted, even among well-known mathematicians, as late as the middle of the 16th-century, when the Renaissance Italian polymath, Cardano recognized, "minus times minus give plus", but still regarded negative numbers as 'fictitious'.



To illustrate, we can start with the flight equation of a tossed stone: $y = 6 + x - x^2$ (it starts off 6 feet up). The roots are +3 and -2. The -2 answer does not fit the real-world situation, but it is still good mathematics. The logic is irrefutable, but the result is emotionally unacceptable. Mathematics does not need to be justified by a real-world situation.
Time always has a direction: Future is (+) and the past is (-). A depreciation in finance is a 'negative appreciation'. Double negative in language: "There is no way I am not going to be there".

3. Fraction

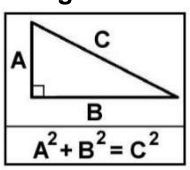


The social obligation to share things, like food, most likely was one of the key drives for humans to create fractions. The language of halves, quarters, thirds, etc.
It has been recognized that children who grew up with digital clocks (eg, smart phone) have more difficulty understanding fractions than with children who grew up with analog clocks (eg, saying “quarter to 6”, “half past 3”).

Fractions can be broken into two categories:
1) Rational Number. A rational number is a number that can be expressed as the fraction A/B of two integers, a numerator A and a denominator B (B can’t be zero). Since B may be equal to 1, every integer is a rational number (7=7/1). A rational fraction can also be written without using explicit numerators or denominators, by using decimals or percent signs (as in 0.01, 1/100, or 1%). The word ‘rational’ gets its name from the Latin *ratio*.

2) Irrational Number. An irrational number is a number that cannot be expressed as a ratio of integers (a fraction). When written as a decimal number, irrationals do not terminate or repeat. The classic example is the square root of 2 ($\sqrt{2}$=1.41421356237…) which shook the Pythagoreans to the roots of their beliefs.

4. Algebraic Number



For the Pythagoreans (6th-century BC), the natural number was at the center of the universe - it was at an emotional, mystical and religious level. Their attitude was that natural numbers had a ‘sacred’ truth since everything was derived from natural numbers. Their belief (ideology?) became a priesthood and the ‘truth’ had to be guarded. The Pythagoreans accepted fractions, for every fraction was a ratio of two natural numbers. The Pythagorean theorem states that sides of every *right* triangle satisfies the formula $a^2 + b^2 = b^2$. Everything was fine until someone looked at the diagonal of a square with side length of 1. The Pythagorean Theorem says that the diagonal length is $\sqrt{2}$ – illogical.

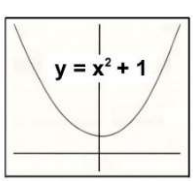
$$\boxed{\sqrt{2}}$$

The discovery that shattered the Pythagoreans was that $\sqrt{2}$ was not a fraction of natural numbers – this was a matter serious beyond belief. The whole school of Pythagoras was sworn to secrecy concerning this dreaded fact. Around 5th-century B.C., Hippasus of Metapontum, a Pythagorean, discovered irrational numbers. According to Greek legend, shocked Pythagoreans had Hippasus drowned at sea for the punishment of 'upsetting' the gods.
Since $\sqrt{2}$ cannot ‘go’ into fractions, algebraic numbers were invented (discussed further in CH7, pg. 18).
Note: In the generalized form of Pythagorean’s theorem ($a^n + b^n = c^n$), Fermat's Last Theorem (1637) states that there are no three positive integers a, b and c that satisfies the equation for any integer value of n greater than 2. After 358 years of effort by mathematicians, the first successful proof was published in 1995 by Andrew Wiles.

5. Imaginary Number

$$\boxed{\sqrt{-1}}$$

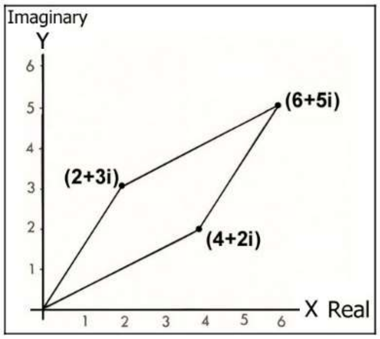
In the ‘Negative Number’ discussion (pg. 4), it was shown that the answer, the ‘roots’, of the stone flight equation had two answers, one positive and one negative. A negative number doesn’t make practical sense in the real world, but its ‘good mathematics’. The same goes for taking the square root of a negative number. It applies to ‘worlds’ we can’t see with the eye. Imaginary numbers are essential for the understanding of electronics, AC power and the atom. A deeper example is the application of imaginary numbers to the space-time continuum within cosmological research (a la Steven Hawking). Big “Woo-Woo” sweepstakes winner, there.



Since $\sqrt{-1}$ cannot be expressed by algebraic equations, ‘imaginary’ numbers were invented (the ‘non-Platonists’ point-of-view, pg. 4). Imaginary numbers are represented with the letter i which equals $\sqrt{-1}$.

The equation $x^2 + 1 = 0$ does not have roots (the graph does not cross the x-axis). By letting $\sqrt{-1} = i$ solves $i^2 = -1$. Creating an 'imaginary' number (i) is intellectually compelling, but an emotionally unacceptable approach; a geometrical analogy, and even a social one, would be helpful.

6. Complex Number



One geometric analogy to explain the complex number is the parallelogram. Parallelograms fit exactly the way that certain physical things add up, like forces (known as 'vector math'). The points of the parallelogram are represented by a number with two parts: a 'real' part (x-axis) and an 'imaginary' part (y-axis). The graph to the left shows a parallelogram represented by the complex numbers: (4+2i) + (2+3i) = (6+5i)

In the physical sciences, forces can be interpreted using complex numbers. Electronic engineers could not operate circuit theory without $\sqrt{-1}$ or i.

Operations Table for 'Travel Commands'

	A	L	U	R
A	A	L	U	R
L	L	U	R	A
U	U	R	A	L
R	R	A	L	U

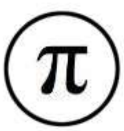
A = Go-Ahead = +1 L = Left = i
U = U-Turn = -1 R = Right = -i

Operations Table for Complex Number

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

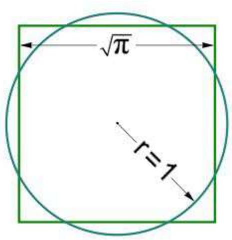
For a 'social' analogy, complex numbers can be related to walking commands.
A = Go-Ahead = +1 U = U-Turn = -1
L = Left-Turn = i R = Right-Turn = -I
The 'truth tables' to the left show how walking commands relate to complex numbers.

7. Transcendental Number



In primitive times the ratio of a circle's circumference to its diameter (C/D) was taken as 3, which was good enough. A simple close fraction is 22/7 (3.14). The Chinese used 355/113 – accurate to 5 digits (3.14159). Since the mid-18th century, the Greek letter π has been used to represent the mathematical constant of the circle's C/D ratio – a number that is infinitely long: 3.141592653589793238462643383279502884197169399375105820974944592307…
Contemplating pi's infinite nature, the mathematician can envision the 'purity' of the circle (also another defense on the 'discovery' of number, pg. 4). On the other hand, the practical engineer sees π as a useful mental construct but knows there's no such thing as a perfect circle in the known physical universe. In 1882, the mathematician Lindeman proved that π was one of a new class of numbers – 'transcendentals' – a number that cannot come from an algebraic equation. Another transcendental constant is e, the base of the natural logarithm, named after the Swiss mathematician Leonhard Euler (CH7, pg. 25).

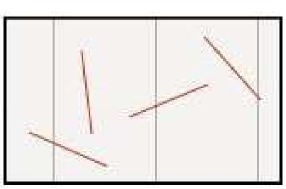
Squaring the Circle



Algebraic numbers come as the solutions of equations with whole number coefficients, but π cannot be determined that way. The Greeks successfully drew squares equal in area to any triangle, but with a circle the problem proved intractable – known as 'Squaring the Circle'. The difficulty was using only a compass and protractor to make area of a square equal to the area of a circle.

The Magical & Mysterious π

Pi is found in many areas where it's relation to the circle is not self-evident. Some say that pi isn't a number, but a "startup constant of spacetime". That the *existence of space* has a numerical signature – called a transcendental number.

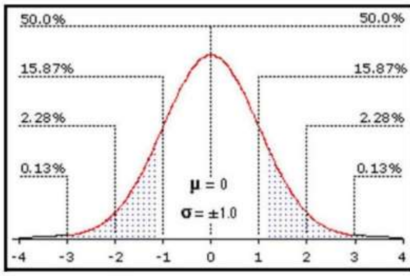


1) Probability & π. In the 18th century, French philosopher Georges-Louis Leclerc, Comte de Buffon determined that you could approximate π by

dropping needles on a grid of parallel lines (spacing is greater than the length of a needle) and calculating the probability that they will cross a line. The probability is directly related to π (search 'Buffon's Needle' for the detailed proof).

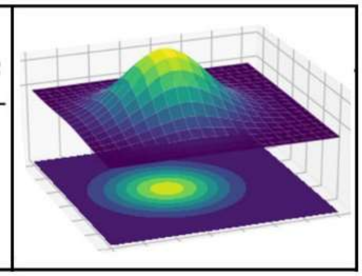
2) River & π. In a paper titled "River Meandering as a Self-Organization Process" (1996), Hans-Henrik Stølum studied the chaotic behavior of a river's form over time. The value of a river's 'sinuosity' – the ratio of the actual length and the direct length as the crow flies - tended to oscillate between a low-value of 2.7 and a high-value of 3.5, but with an average sinuosity of 3.14. Stølum justified the result using fractal geometry.

3) Normal Distribution & π.



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{(x-\mu)^2}{-2\sigma^2}}$$

Probability Density Function (PDF)



Normal distributions are important in statistics and are often used in the natural and social sciences to represent real-valued random variables whose distributions are not known (eg, height & weights of men and women). Pi shows up in the normal distribution since the integral of the Probability Density Function (PDF) must sum to one. Pi is the factor that makes the area under the curve equal to one (relates to a circle with area of 1) . For a visual explanation the projection of a 3D PDF distribution on to a 2D surface will yeild concentric circles (the 'shadow' of the distribution).

Other transecendental numbers are trigometic functions (CH7, pg. 24) and the natural exponential function (CH8, pg. 28).

The Most Beautiful Equation in the World

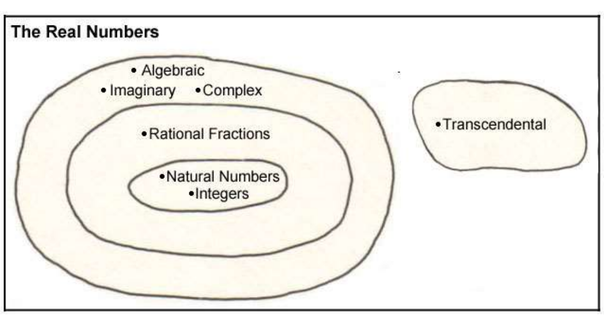
For reasons that seem mysterious, let x = πi. With some 'pencil whipping', $e^{\pi i}$ has a very special result.

$\boxed{e^{\pi i} + 1 = 0}$

For some mathematicians, the linking of e, π, i, 1 and 0 in this manner is a truly moving experience ('mystical' to some). Mathematical solutions to real life problems come out in terms of the exponential function e^x (CH8, pg. 26).

The electrical engineer demystifies the above 'mystery' equation as $e^{\pi i} = -1$. In this form, it merely states that a rotation by π radians (180 degrees) is simply a reflection or multiplication by -1.

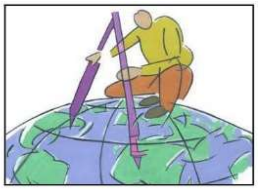
8. Real Number



In concluding the discussion on 'Numbers', real numbers comprise all the previous numbers:

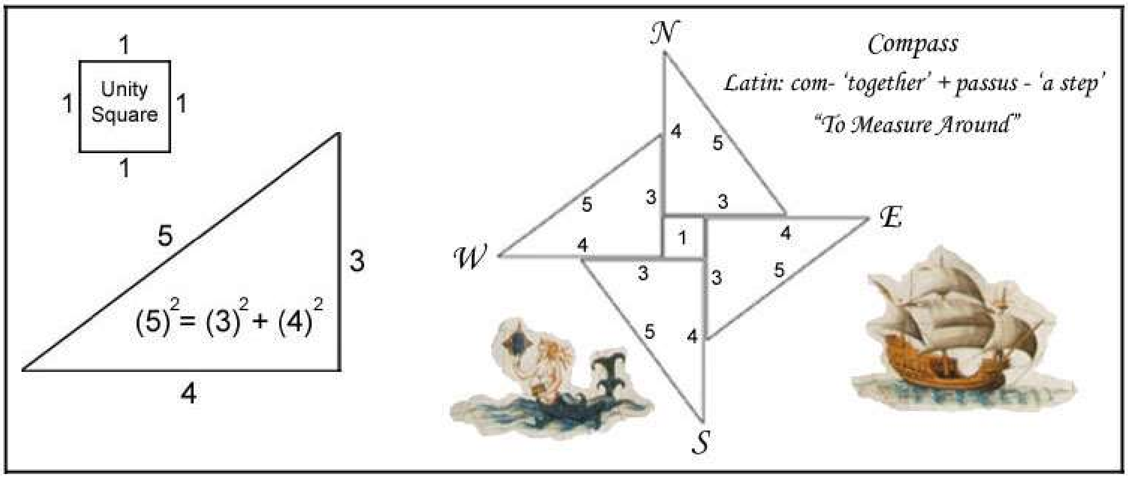
Notice that the Transcendental set is outside the main set – that's because transcendental numbers cannot be the roots of an algebraic equation.

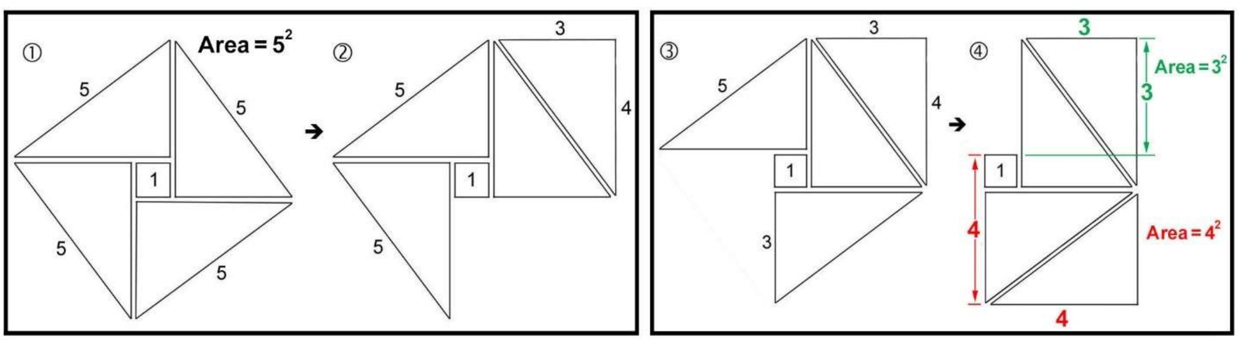
CH6 – GEOMETRY



Geometry literally means 'earth measuring'. The builders of ancient Egypt, thousands of years before Pythagoras, knew that if they used a knotted rope, with 12 equal gaps between the knots, then when the rope was formed into a 3-4-5 triangle there was a right angle opposite the largest side. It was Pythagoras who later figured out the square relationship: $3^2 + 4^2 = 5^2$. The term 'square' (and even 'cubed') is not just a mathematical term – it comes from our description of a four-sided object with equal sides – the square.

The Pythagorean Theorem can be understood visually using the 'Unity Square' and four '3-4-5' Pythagorean triangles. You first start with a square of area of 25 (5^2). The goal is to rearrange the triangles so that we get two squares - one with and area of 9 (3^2) and the other with an area of 16 (4^2). The areas of ① and ④ are the same – a visual proof of the Pythagorean Theorem.



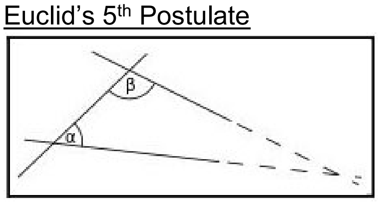


Euclidean Geometry

Classic geometry focused on compass and straightedge constructions. Geometry was revolutionized by Euclid (~300BCE) – the 'father of geometry' - who introduced mathematical rigor and the axiomatic method still in use today. Euclid's highly influential 'Elements' consists of 13 books - a collection of definitions, postulates (axioms) and proofs of propositions.

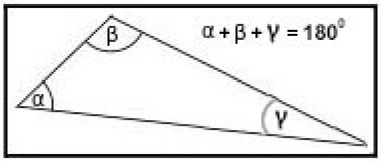
Before geometry was even thought of, man stretched ropes on the ground and made triangles. The Greeks abstracted the 'rope triangles' and built an ideal geometrical world with a thing called a straight line in that was "the shortest distance between two points".

Euclid's 5th Postulate



The 'Parallel Postulate'. "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

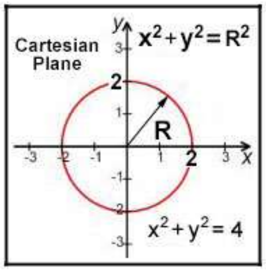
Triangle Angle Sum Theorem



The sum of the three angles of any triangle will always equal 180 degrees.

Newtonian mechanics and the universal system of gravitation assumed a Euclidean geometry.

Cartesian Coordinate System

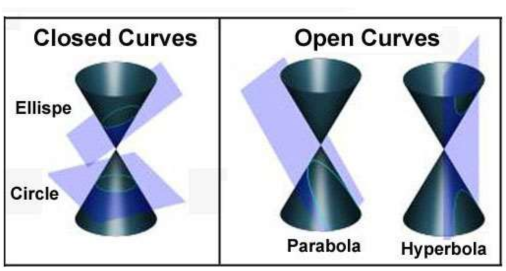


Rene Descartes (1596-1650) developed ‘Cartesian’ or analytic geometry, which uses algebra to describe geometry. The Cartesian Coordinate System (CCS) enables a geometric shape or curve to be described by using an equation rather than an elaborate geometrical construction. This enabled a formal distinction between curves that can be defined using algebraic equations, *algebraic curves*, and those that cannot, *transcendental curves*.

For example, a circle of radius 2, centered at the origin, is described by the equation $x^2 + y^2 = 4$. In CCS space this is the number line in two dimensions, known as the Cartesian plane. In Euclidean space, the Cartesian x and y axis are perpendicular to each other. If not, then non-Euclidean geometry is required.

The Conic Sections

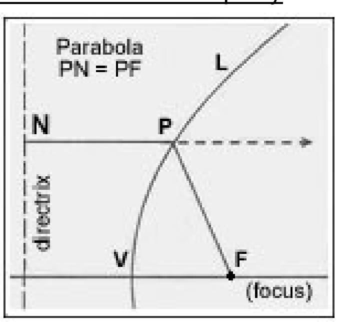
The Greeks used the Conic Section to generate four basic types of geometric shapes: circles, ellipses, hyperbolas and parabolas. A conic section is the intersection of a plane and a double circular cone. By changing the angle and location of the plane’s intersection different types of conics are produced.



Closed Curves
1) Circle: plane slice is 90° to the cone’s axis.
2) Ellipse (‘oval’): plane slice is tilted between 0° & 90°.

Open Curves
1) Parabola: plane slice is parallel to the edge of the cone.
2) Hyperbola: plane slice is parallel to the cone’s axis (0°).

Focus-Directrix Property

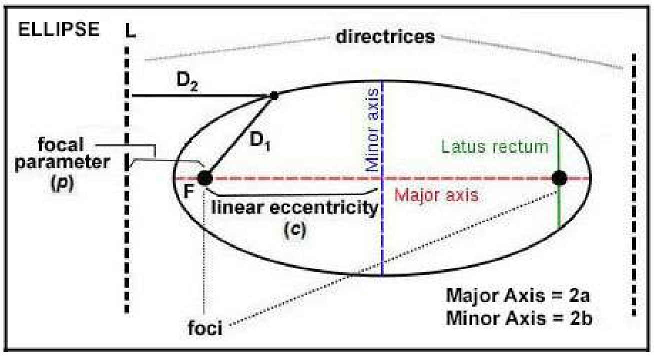


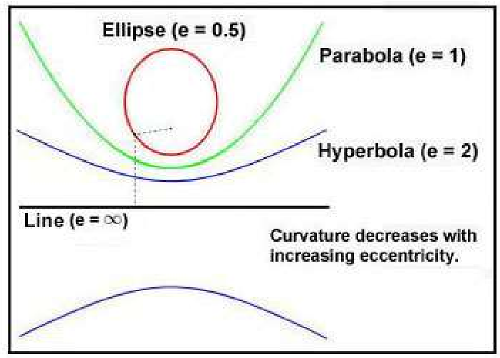
As an alternative to Conic Sections, the Greeks discovered a formal method, known as the ‘focus-directrix’ property, to generate circles, ellipses, parabolas and hyperbolas. The figure on the left illustrates the process:
1) fixed line: the directrix.
2) fixed point: the focus (F)
3) moving point: P
4) Point P moves where its distance from the directrix and from the focus (F) stay in proportion.

# of Focuses	Curve Type
1	Parabola (NP = NF), Ellipse (NP = 2PF)
2	Ellipse, Hyperbola

Conic Section Equations

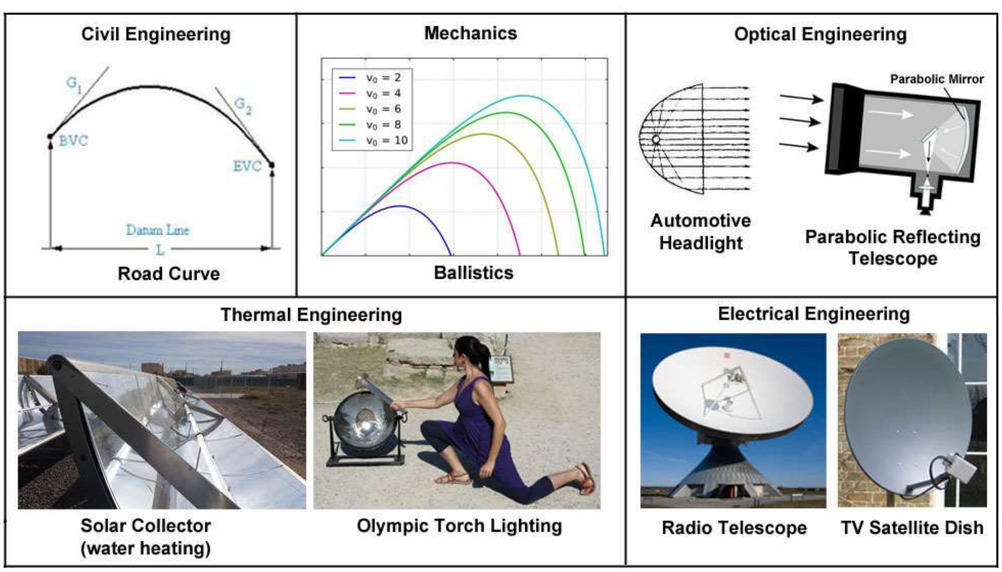
conic section	equation	eccentricity (e)	linear eccentricity (c)	semi-latus rectum (ℓ)	focal parameter (p)
Circle	$x^2 + y^2 = a^2$	0 b = a	0	a	∞
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$ 0 < e < 1 b < a	$\sqrt{a^2 - b^2}$ > 1	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
Parabola	$y^2 = 4ax$	1 b = 0	a	$2a$	$2a$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$ e >1 b > a	$\sqrt{a^2 + b^2}$ > 1	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$



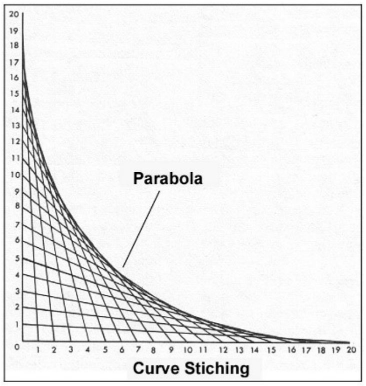


The Practical Parabola

Engineering Discipline	Parabolic Applications
Civil	1) Vertical Curves for roads - parabolic cross-section makes for a smooth ride over the ‘hump’ (geomatics).
Structural	1) The curves of ‘modern’ bldg. roofs or exotic car roofs (see ‘Curve-Stitching – The Amazing Parabola’)
Mechanics	1)The path that a thrown or ballistic projectile takes under the force of gravity.
Thermal	1) Thermal Collector (solar water heater). 2) Torch lighting with the light of the Sun.
Electrical	1) Microwave Parabolic Antenna. 2) TV Satellite Dish
Optical	1) Automotive Headlight. A source of light at the focus of a parabolic mirror produces light rays coming out parallel to the axis. 2) Reflecting Telescope. Developed by Sir Isaac Newton (1668) - also known as the "Newtonian Telescope".



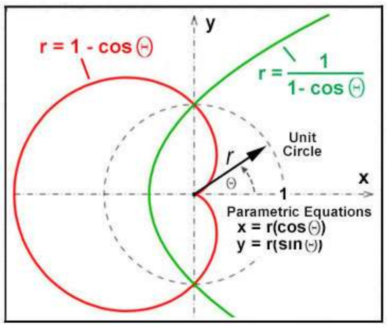
Curve-Stitching - The Amazing Parabola



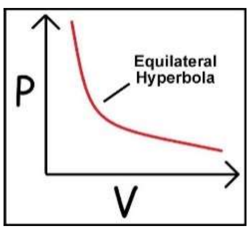
To teach children about number relationships there is a common activity know as curve-stitching. For example, the task is to determine what pairs of numbers add to 20: 0+20, 1+19, 2+18, etc. For a visual demonstration lines can be drawn from each number pair. The final result is surprising – a parabolic curve appears.

A real-world example is seen in modern architecture – the shape of beautifully shaped roofs are often hyperbolic paraboloids. One sectional view is parabolic and the other sectional view is hyperbolic.

Cardioid – Inverse Parabola



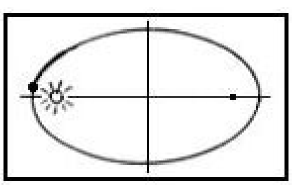
A cardioid is the inverse curve of a parabola with its focus at the center of inversion (Fractal Geometry, pg. 13).



Curves for any xy product that is equal to a constant result in equilateral hyperbolas. For example, equilateral hyperbolas are seen in Boyle's Law where the product of pressure and volume is a constant (assuming constant temperature and constant mass).

Celestial Mechanics

With a vast number of observations and some monstrously heavy calculations, Kepler found three laws of planetary motion. One law informed that the planets travelled in an ellipse and another law established the speed of their paths.

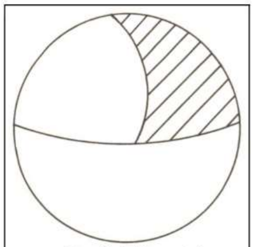


Kepler's Laws

1. All planets move in elliptical orbits with the sun at one focus.
2. A line joining any planet to the sun sweeps out equal areas in equal times.
3. The square of the period of any planet about the sun is proportional to the cube of the planet's mean distance from the sun.

Later on, Newton discovered the universal law of gravitation which asserted that celestial bodies (Earth, Moon, Sun, etc.) pulled on every other on with a force depending on their masses and the distances apart. The end result is that planets, comets, satellites all had to travel in one of the conic sections. Halley's Comet travels in a very long flat ellipse. Other comets move through on a hyperbola and disappear off into space. To travel in a circle or a parabola is a very special case, if ever. typically, a celestial path is an ellipse or a hyperbola.

Non-Euclidean Geometry

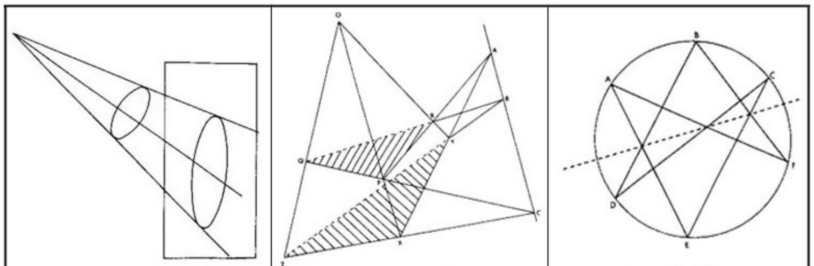


Earth 'Triangle'

Euclidean geometry breaks down when a triangle is constructed on the face of the Earth – the fixed sum does not work. For example, a triangle starting from the North Pole down to the Greenwich Meridian to the Equator and back to the North Pole. All three angles are 90°, but total 270° – these are great circles, they are not straight lines.

In the geometry of a sphere's surface, a line cannot be drawn line parallel to a given line through a given point – non-Euclidean geometry.

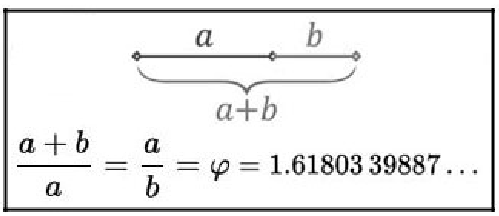
Projective theorems: 1) Desargues, 2) Pascal.



Projecting a Circle | Desargues' Theorem | Pascal's Theorem

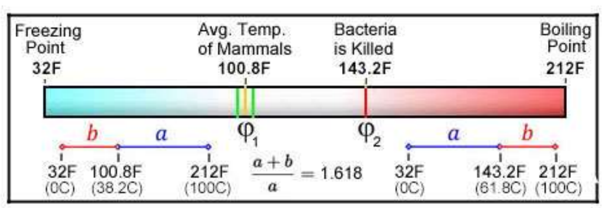
If the 'Euclidean' geometrical equations were drawn on a rubber balloon, the lines become distorted as the balloon grows – another field of mathematics called 'topology'.

The Golden Ratio



The golden ratio, or divine proportion, is a number often encountered when taking the ratios of distances in simple geometric figures. It an irrational number meaning it cannot be written as a simple fraction. Two quantities a and b are said to be in the golden ratio if the ratio of (a+b)/a is equal to the ratio a/b. Psi, the symbol for the Golden Ratio, can be easily calculated using the quadratic formula.

Golden Mean ratios appear everywhere in the universe. The spiral is the natural pattern of water flowing drown a drain; the flow of air in tornados and hurricanes; the spiral form of the Nautilus shell; the number of lines in the spirals of a sunflower; the spiral of a galaxy. The Golden Mean ratio is also all over the human body: the ratios between bones and the length of your arms and legs; the ratio in the distance from the navel to your toe and the distance from your navel to the top of your head.



Another fascinating example of finding the Golden Ratio in nature is in the spectrum of mammal body temperatures and the temperature at which bacteria is killed. Using the temperature end points of freezing and boiling, the Golden Ratio appears. Uncanny.

Fractal Geometry

"… no one, not even Benoit Mandelbrot himself…had any real preconception of the [Mandelbrot] set's extraordinary richness. The Mandelbrot set was certainly no invention of any human mind. The set is just objectively there in the mathematics itself. If it has meaning to assign an actual existence to the Mandelbrot set, then that existence is not within our mind, for no one can fully comprehend the set's endless variety and unlimited complication." — Roger Penrose (The Road to Reality)

Being that fractals are not typical geometric objects, understanding them involve algebra, calculus and infinite series (CH7, CH8). A fractal is a mathematical set that exhibits a repeating pattern displayed at *every scale*. Fractals are different from other geometric figures because of the way they scale, sometimes described as "self-similarity'. Fractals are an expanding geometric symmetry where the whole of the fractal looks just like a part of the fractal at successive sub-levels. Technically, self-similarity is an informal definition. A formal definition of a fractal is actually quite difficult, a bit slippery and if fact there is no widely accepted formal definition.

Approximate fractals found in nature that display self-similarity include: river networks, mountain ranges, earthquakes, craters, lighting bolts, coastlines, algae, trees, pineapples, snowflakes, crystals, ocean waves, heart rates, blood vessels, proteins and even the rings of Saturn.

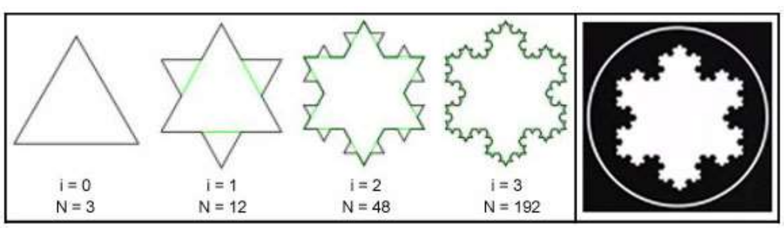
In Chapter 5, "The Great Math Mystery" (Number, pg. 4), speculated on whether math is discovered or invented. The Platonic school of thought says that math is discovered. Roger Penrose, a mathematical Platonist, believes that a fractal pattern is not a mental construct, but has its own existence on a Platonic plane waiting to be discovered.

Mathematical 'Monsters'

The story of fractals begins in the late 19th century with mathematicians attempting to create mathematical curves of objects found in nature – clouds, flower, plants. The kind of formulas they came up with satisfied the definition of a curve, but they were not lines or circles, just very peculiar. There were so weird that they were beyond what could be drawn. These unfamiliar concepts were sometimes referred to as mathematical 'monsters': a 1-dimensional line yet having a dimension that resembles a surface; a continuous curve, but not differentiable (analytical functions are defined as continuous and infinitely differentiable) – a paradox.

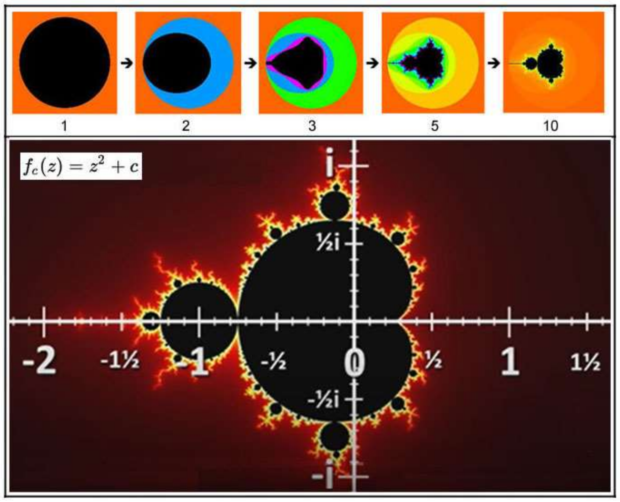
Koch Snowflake

A Koch snowflake (1904) is a fractal that begins with an equilateral triangle and then replaces the middle third of every line segment with a pair of line segments that form an equilateral 'bump'. The number of sides is $N = 3 \times 4^i$, where i is the number of iterations. Since the number of sides are predicable, the Koch Snowflake is a 'deterministic' fractal.



The Koch snowflake (curve) is a paradox – the surface it appears to be finite, but mathematically it is infinite. Theoretically, it cannot be measured. It's known as a 'pathological' curve – it conflicts with the rules of Euclidian space and measurement. With infinite repetitions we get an infinite perimeter, however the area is not infinite – put a Koch flake inside a circle, it's always inside.

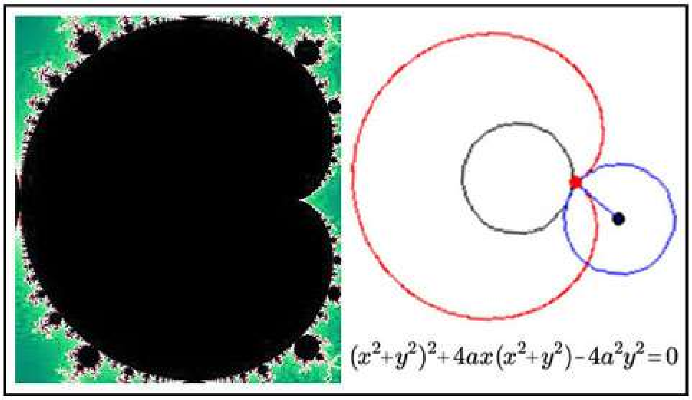
Mandelbrot Set



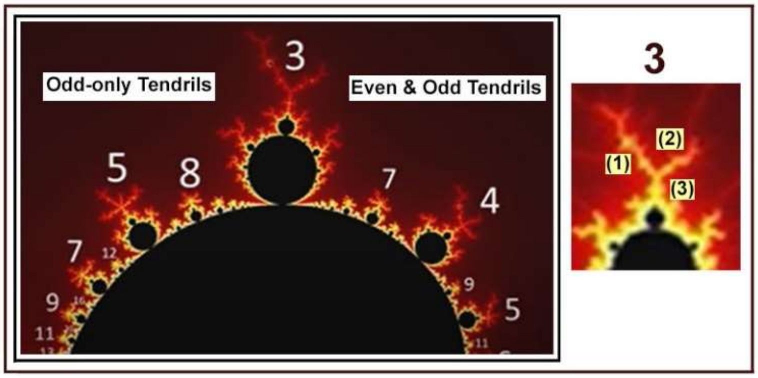
In the early 1960s, mathematician Benoit Mandelbrot used the term 'fractal' in papers about self-similarity and in studies about determining the length of a coast (eg, "How Long is the Coast of Britain?"). Computers made it easy to do the vast iterations demanded by the 'math monster'. Computers were essential to unlock the inner secrets of fractals. In 1980, Mandelbrot created a set of fractals based on the equation: $f(z) = z^2 + c$. The output of each iteration was feedback into the input for the next iteration. The equation became the emblem of fractal geometry – the Mandelbrot Set.

The Mandelbrot sequence at the top of the above figure (1, 2, 3, 5…10) is based on a static number of iterations per pixel for a set of complex numbers c. By iteration 10, the classic shape is present ('non-deterministic' fractal). The black areas represent values that are members of the Mandelbrot set whereas deep red areas represent strict non-membership. The red-to-yellow gradient signifies boundary membership values.

A Deep Dive into the 'Monster'

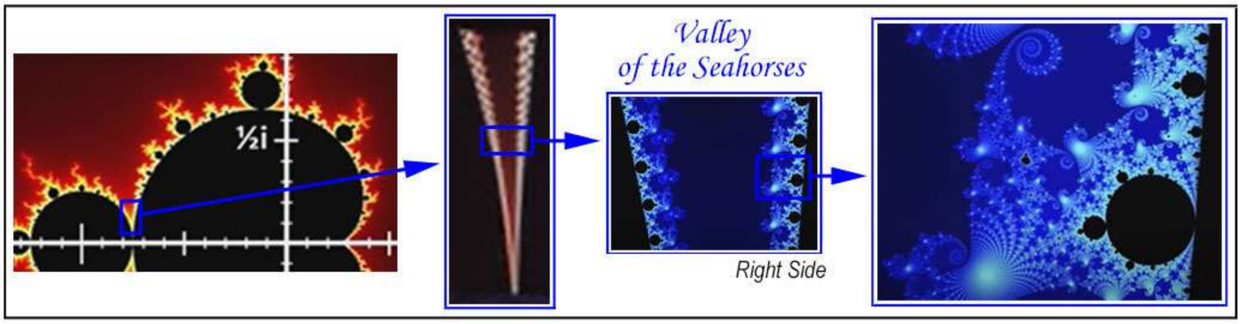


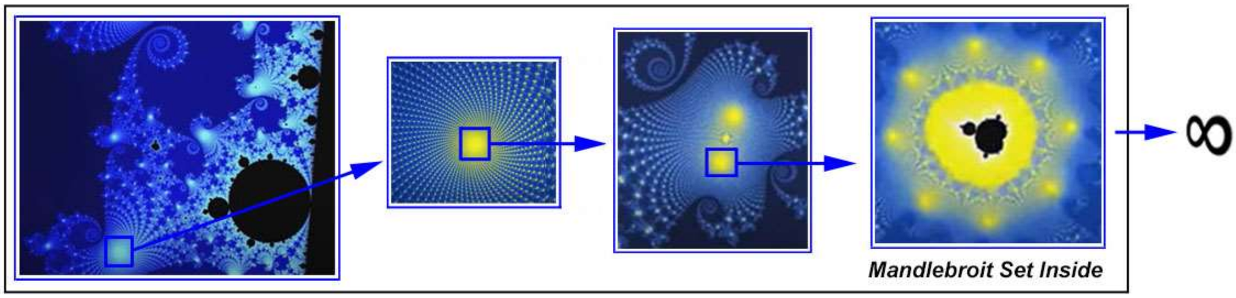
The 'heart' of the Mandelbrot Set is a perfect cardioid which is a curve that can be traced out by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius. A cardioid is also the inverse curve of a parabola with its focus at the center of inversion ('Practical Parabola', pg. 10).



'Tendrils' project out around the small black circles that circumvent the cardioid. The top circle has three tendril branches. On its left side all the black circles have an odd number of tendrils. On the right-side, the tendril count is and even and odd.

Valley of the Seahorses



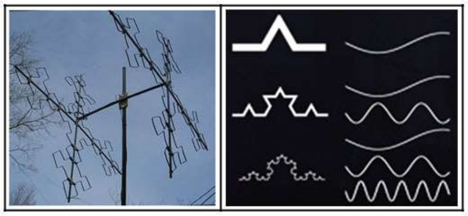


Valley of the Double Spirals

Fractals in Nature

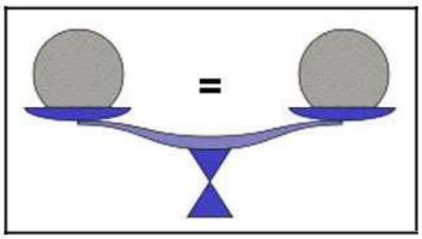
Fractal Applications

Fractal research has been applied to many technological and scientific disciplines including: heat exchangers, digital imaging, urban growth, enzyme research, signal & image compression, computer graphics and game design, fracture mechanics, T-shirts and other fashion, camouflage, digital sundial, networks, medicine, neuroscience, diagnostic imaging, pathology, geology, archaeology, soil mechanics, seismology, financial market analysis and antenna design.



Nathan Cohen attended an astronomy conference where Dr. Mandelbrot discussed the large scale of the universe and the potential of fractals for being good research tools to improve structure research. Cohen, being a hobbyist ham operator, thought about making an antenna in the shape of a fractal. To test his idea, he built a simple snowflake antenna for his ham radio. The antenna worked and it took up less than a quarter of the space of a traditional antenna. Cohen discovered that the fractal antenna was receptive to many frequencies. Self-similarity was the key. He and others later demonstrated mathematically that the fractal approach was an optimum way in antenna design (using Maxwell's electromagnetic equations). Fractal antennas are now used in most cell phones.

CH7 – ALGEBRA



In algebra we are concerned with equating like with like (the original Arabic word meant the reunion of broken parts or restoring to normal). There is always a risk using an analogy since analogies seldom fit exactly. Put on a balance two identical balls – they balance out. Replace one of the balls with shoes. Balance can also be achieved. Way to explain the Identity statement: (a + b) = (b + a) and working with non-identical items.

<u>Math rules for multiplication</u>
Commutative: ab = ba
Associative: a(bc) = (ab)c
Distributive: a(b+c) = ab + ac

Note: symbolism is being used; "ab" means a x b. Let no one try to tell you that mathematics, and particularly its symbolism is clear, concise and unambiguous. It is a mess.

We have had to accept English spelling. To illustrate the mess, look at these conventional notations:

2 ½ is 2 + 1/2 25 is (2 x 10) + 5 2a is 2 x a

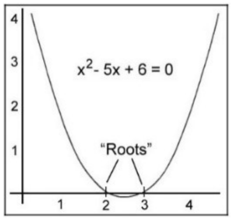
A symbol has been written by the numeral 2. In every case 'is' means something quite different.

The distributive rule expresses how addition and multiplication related one to another. The rules for subtraction and division are basically a reverse or inverse process.

In traditional algebra we learned at school, we looked at expressions with letters in them and tried to figure out how to rearrange them, such as:
3a + 2b +4a + 6c - 2a + c – b = a(3+4-2) + b(2-1) + c(6+1) = 5a + b + 7c
It was slog work. Endless hours were spent in the exercise, requiring practice, but nothing creative or really thoughtful.

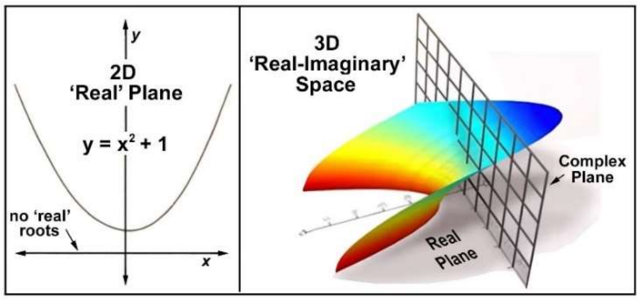
Some expressions are called formulae - usually when they express one particular result. When Galileo built a ramp to study the motion of falling objects, he discovered that distance is proportional to time squared – the Law of a falling body.
The formula for the distance is: $x = V_0t + 1/2at^2$. As an example, let's calculate the distance a stone falls after 3 seconds. The initial velocity (V_0) is 0 and the acceleration (a) due to gravity is $32 ft/sec^2$. Plugging in the parameters and variables into the distance formula: $x = 0 + ½(32)(3^2) = 16 \times 9 = 144$ feet.

Fundamental Theorem of Algebra



Striped to its essence, the fundamental theorem of algebra states that a single-variable, degree n polynomial has exactly n roots. A degree 1 polynomial has one root (or solution) - known as 'linear' equation. A degree 2 polynomial has two roots whose characteristic is a curve (quadratic). A degree 3 polynomial, having three roots, is also a curve, but with a faster rise.
The equation to the left (x^2-5x+6=0) has the two roots x=2 and x=3 (the parabola intersects the x-axis twice).

Imaginary Numbers & 3-D Space



In Chapter 5, Imaginary Number (pg. 5), the equation x2 + 1 = 0 was shown to have no roots (the graph does not cross the x-axis). The traditional x-y cartesian graph is unable to show the 'imaginary' roots (i, -i) since it is a two-dimensional graph.

The imaginary roots are revealed with a 3-D graph which implies that the solutions to the "x2 + 1 = 0" equation is a 3-D shape (!). The imaginary plane points upwards from the 'real' plane x-axis. It is in this plane where the imaginary roots are located (for a complete explanation visit a math tutorial on the Internet).

Linear Algebra

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
2 x 2 Matrix

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
3 x 3 Matrix

$$|A| = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$
4 x 4 Matrix

There are many applications that involve multiple linear equations of degree 1 with more than one variable – a branch of mathematics known as 'linear algebra'. The basic rule in solving linear equations is that the number of equations must be no less than the number of variables. Suppose we are to find the solutions of three equations with three unknowns (x, y, z). The brute force method would be to use one of the equations and to solve for one of the variables (x = ay + bz) and then to plug that variable (x) into one of the other two equations and to repeat the process until the three variables are solved for. Linear Algebra simplifies this process with a set of n x n matrix rules. The example below illustrates the 'elimination method' to solve for x, y, z from three equations. The goal is to produce one row with one zero and another row with two zeros (done by row multiplication and addition/subtraction).

Equations into Matrix

$$\left.\begin{aligned} 2x + y - 3z &= -4 \\ 4x - 2y + z &= 9 \\ 3x + 5y - 2z &= 5 \end{aligned}\right\} \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 4 & -2 & 1 & 9 \\ 3 & 5 & -2 & 5 \end{array}\right]$$

Matrix Goal

$$\left[\begin{array}{ccc:c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

New Row 2 = -2 x (Row1) + 1 x (Row2)

$$\xrightarrow{(2)} \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 0 & -4 & 7 & 17 \\ 0 & 7 & 5 & 22 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 2 & 1 & -3 & -4 \\ 0 & -4 & 7 & 17 \\ 0 & 0 & 69 & 207 \end{array}\right] \rightarrow \left[\begin{aligned} 2x + y - 3z &= -4 \\ -4y + 7z &= 17 \\ 69z &= 207 \end{aligned}\right] \begin{aligned} &\rightarrow x = 2 \\ &\rightarrow y = 1 \\ &\rightarrow \boxed{z = 3} \end{aligned}$$

There are problems, such as airflow past an aircraft wing or water past the supports of a bridge that are quite complex if the shape is not simple. Transformations can be used in such problems to reduce the difficult solutions to an easier one. It can be quite sophisticated, but the principle is to transform the problem to one that can be tackled, and a matrix may be one of the techniques for doing so. The ramifications of matrices are considerable and have the ability to solve practical applications that lie beyond the everyday.

Linear Algebra behind Google

$$\begin{bmatrix} 0.0375 & 0.0375 & 0.8875 & 0.4625 \\ 0.3208\bar{3} & 0.0375 & 0.0375 & 0.0375 \\ 0.3208\bar{3} & 0.4625 & 0.0375 & 0.4625 \\ 0.3208\bar{3} & 0.4625 & 0.0375 & 0.0375 \end{bmatrix}$$

The speed of Google's ability to search for web pages is based on its patented PageRank Algorithm which quantitatively rates the importance of each page on the web.

The ranking process is based on standard linear algebra (plus some other 'magic' math). For complete details, search the article, "The $25B Eigenvector - The Linear Algebra Behind Google".

Polynomials

In Chapter 5 (Numbers) algebraic numbers began with the problem of taking the square root of 2 since it could not be expressed as a ratio of integers (A/B). When the rules of math break down, a new branch of mathematics opens up. The square of 2 problem opened the mathematical branch known as the Polynomials. The equation to solve $\sqrt{2}$ is: $x^2 - 2 = 0$. But that is the particular answer. The general answer became the *finite* polynomial function, which is an expression built from *whole* number constants and symbolic operators (additional, multiplication & exponentials). Formally, the polynomial is written as:

$$\boxed{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0}$$

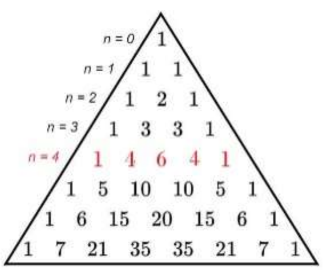
Power Series

Ok, now were starting to look like 'real' math – messy and complicated looking. The above general polynomial is 'cleaned up' using the summation notation, which turns the polynomial function into:

$$\boxed{\sum_{i=0}^{n} a_i x^i}$$

Better – kind of beautiful. When 'n' is infinite, it's known as the 'Power Series'. (an advanced expression is the 'Taylor Series', pg. 29).

Binomial Theorem

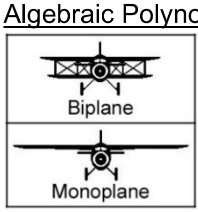


The binomial theorem expands a two-term polynomial $(x + y)^n$ as:

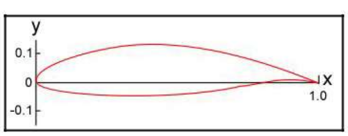
$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}} \quad \boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

where the constants n and k are whole numbers. The Binomial is the simplest kind of polynomial. For example, if n = 4: $(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$. The binomial coefficient for varying n and can be arranged to form Pascal's Triangle.

Algebraic Polynomial – Real World Application



Early 20th-century airplanes were known as biplanes due to wings stacked one above another. It was the era of wood and cloth construction where the bi-wings were braced with wires and struts - similar to the mechanical design of truss bridges. Advances in engine power and aluminum manufacturing in the 1930's ushered in a new era in aviation - the metal cantilever monoplane. Stronger metallic wings removed the need for multi-sets of wings and external bracing. Previous airfoil design was largely based on empirical 'trial & error' efforts. To improve the lifting ability of the new metallic wings, intensive formal research was put into airfoil design, where 'formal' means *mathematical*.

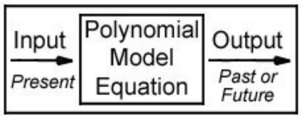


Aeronautical researchers used the polynomial function to mathematically describe the shape of an airfoil – a boon for wind tunnel research and for consistent, quality manufacturing. In the beginning a four-digit code was used to define the wing profile (eg, NACA 2412). Part of digits related to coefficients of the polynomial function. A sample of an airfoil algebraic polynomial looks like:

$$\boxed{y = 5tc\left[0.2969\sqrt{\frac{x}{c}} + (-0.1260)\left(\frac{x}{c}\right) + (-0.3516)\left(\frac{x}{c}\right)^2 + 0.2843\left(\frac{x}{c}\right)^3 + (-0.1015)\left(\frac{x}{c}\right)^4\right]}$$

If that appears messy and complicated – it is. What happened to the integer coefficients? Welcome to the real world of engineering mathematics. The trick is to multiply all the terms of the airfoil function by a specific (large) number to make all the decimals turn into whole numbers.

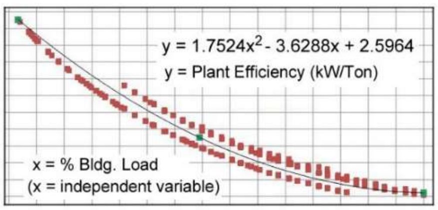
Algebraic Polynomial – Modeling: Predicting the Past & Future



Algebraic polynomial equations are commonly used to mathematically model a system to predict conditions of the past or the future. The 'system' widely varies – financial markets (derivatives), environmental systems (global climate change), physics (computational physics), astrophysics, climatology, chemistry, biology and human systems in economics, psychology, social science, and engineering.

For an engineering example, a predictive model can be created to determine the efficiency of an office building's cooling plant. An engineering firm offers an energy-savings system to lower the cost of generating chilled (cold) water for the building's air-conditioning units. After the system is installed, the management is going to wonder if the new system is really saving energy and money. The present cost of running the plant is known (from the utility bills), but a model is needed to predict what the energy and utility cost would be *without* the new system – a model to predict past conditions.

Algebraic Polynomial – Numerical Analysis



One approach is to sample the electrical energy (kWh) of the various components (chiller, pumps, fans) for a specific period of time. The longer the sampling period, the better the predicative model or equation. Modern spreadsheets, like MS-Excel, can perform numerical analysis on the energy data and calculate the polynomial modeling equation. The chart to the left shows one possible modeling equation for a cooling plant.

This is a simplistic example, for quite often in real world applications several separate modeling equations are required to predict the system behavior. This is known as "piecewise" modeling, where each 'piece' equation is determined by a specific range of input data (mathematicians call it 'boundary conditions').

Sequence

In mathematics a sequence is collection of objects in which repetitions are allowed. Sequences can be finite or infinite (eg, the sequence of all even positive integers). In computing and computer science sequences are sometimes called strings, words or lists (infinite sequences are called streams). The convergence properties of sequences are useful in the study of functions, spaces and other mathematical structures. Sequence is the basis for the mathematical series – important in differential equations and analysis.

Fibonacci Sequence

In 1202 Leonardo of Pisa introduced the sequence of Fibonacci numbers. Each number in the sequence is the sum of the two preceding numbers starting with the root number one: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc. While the number of Fibonacci numbers is infinite, sequence is not an 'infinite series', as discussed earlier – no operations are performed on the terms. Interestingly, the ratio of two adjacent Fibonacci numbers progresses towards the Golden Mean (144/89 = 1.618). The Fibonacci sequence and Golden Mean ratios appear everywhere in the universe. The spiral is the natural pattern of water flowing drown a drain; the flow of air in tornados and hurricanes; the spiral form of the Nautilus shell; the number of lines in the spirals of a sunflower; the spiral of a galaxy. The Golden Mean ratio is also all over the human body: the ratios between bones and the length of your arms and legs; the ratio in the distance from the navel to your toe and the distance from your navel to the top of your head.

Mathematical Series

Infinite Series	
$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$	Divergent $\rightarrow \infty$
	Convergent $\rightarrow$ a value

A mathematical series is the sum of the terms of a sequence. Mathematicians use summation notation to express the series compactly. An infinite series consists of an infinite amount terms where the sum, 'in the limit', is either infinite ('divergent') or a value ('convergent').

Arithmetic Series

$$\text{Finite Series} \quad (M = \text{constant} = \$50/\text{mo.})$$
$$\sum_{i=1}^{12} M = \underset{1}{\$50} + \underset{2}{\$50} + \underset{3}{\$50} + \cdots + \underset{12}{\$50} = \$600$$

A practical example of a *finite* series is a bank loan where a monthly amount is paid for a finite period. The example to the left illustrates a \$50 per month payment for 1 year (interest not considered for simplicity).

When there is a constant difference between each consecutive term (ie, \$50) then the sequence is known as an arithmetic sequence or series when the variables are typically *independent* of each other. The arithmetic mean (or average) of a series of numbers is the sum of all the numbers divided by the total numbers (\$600/12 = \$50).

Geometric Series

$$\text{Finite Series} \quad (10\%\ \text{compound interest})$$
$$\sum_{i=1}^{12} M = \underset{1}{\$50.00} + \underset{2}{\$55.00} + \underset{3}{\$60.50} + \ldots + \underset{12}{\$142.66} = \$1{,}069.21$$

The example to the left illustrates a savings account with 10% compounded interest. The initial amount is \$50 and after 1 year the amount is \$1,069.21.

When there is a constant ratio between each consecutive term (ie, 55/50 = 1.1) then the sequence is known as a geometric series where the variables are typically *dependent* on each other (like the return on investment over a period of time). When there is a compounding effect in a series the geometric mean is more appropriate method to determine the 'average'. The geometric mean is n^{th} root of the product of n numbers. For two numbers, the geometric mean = $(a \times b)^{1/2}$.

Arithmetic Mean	$A = \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{a_1 + a_2 + \ldots + a_n}{n}$
Geometric Mean	$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \sqrt{x_1 x_2 \ldots x_n n}$

To illustrate the importance in selecting the geometric mean over the arithmetic mean, consider the return on investment for an amount of \$100 over 2 years. Suppose the 1^{st} year return was -50% and the 2^{nd} year return was +50%. The arithmetic mean = (-50%+50%)/2 = 0%. But that gives the wrong impression that the investor is breaking even on the investment (no loss or profit). For the 1^{st} year, after a 50% loss, the principle is \$50. For the 2^{nd} year, after a 50% gain, the principle is \$50 x 1.5 = \$75. Using the geometric mean provides a formal way to determine the annual rate of return (to account for negative interest, 1 is added to the percentage): $[(1-0.5) \times (1+.5)]^{1/2} - 1 = -0.134$ or -13.4% annual return. 1^{st} year principle = \$100 x (1-0.134) = \$86.60. 2^{nd} year principle = \$86.60 x (1-0.134) = \$75.00.

Geometric Series: Estimate by Bounding

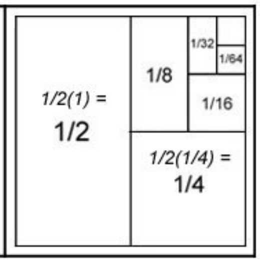
This review of mathematics is largely concerned with *calculation*, however *estimation* plays an important role when we don't need a precise result. There are times where we just need to determine a reasonable result to decide on an action, like 'too big', 'too small' or 'just right'. Greater level of precision requires precise mathematical equations and data (and typically some assumptions or limitations). The consequence is that precise answers take more time. Many questions do not need persnickety precision.

It is often easier and more reliable to estimate upper and lower limits for something than to estimate the quantity directly. For example, to estimate the amount of time each day the average person spends talking or texting on a cell phone. A reasonable lower bound is 2 minutes and an upper bound is 3 hours or 200 minutes. Taking the arithmetic average would be (2 + 200)/2 = 101 minutes, but that gives an estimate that is a factor of 2 lower than the upper limit (200/101 = 2), but a factor of 50 greater than the lower limit (101/2 = 50). The typical goal of an 'estimate' is a result that's within a factor of 10, otherwise we're way off (think of a dart board analogy – a bad estimate doesn't even hit the board). The solution is to take the geometric mean. Going back to the 'average' phone minutes, the geometric mean = $(2 \times 200)^{1/2}$ = 20 minutes. Now the estimate is a factor of 10 lower than the upper limit (200/20 = 10) and a factor of 10 greater than the lower limit (20/2 = 10).

Another example of 'Estimate by Bounding', is the problem of estimating the number of pianos owned in the city of Los Angeles. The ownership lower bound is 1% of the population and the upper bound is 10%. The geometric mean is $(1 \times 10)^{1/2}$ = 2-3%.

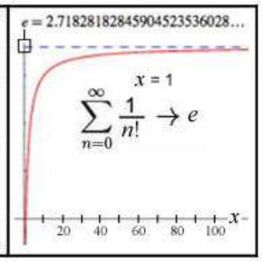
To estimate the population of Los Angeles, a large city, the lower bound is 1 million and the upper bound is 300 million (estimate of the US population; sometimes basic knowledge required). So, the population geometric mean = $(1\text{x}10^6 \text{ x } 300\text{x}10^6)^{1/2}$ = 17 million[1]. So, the piano ownership estimate for Los Angeles is between 1/3 (340K) and half a million (500K) pianos. Remember, the goal is to get an estimate within a factor of 10. So even if we estimated the US population at 100 million the estimate would still be in the ballpark.

Series:	Geometric
Type:	Convergent
Sequence:	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$
Summation: Notation	$\sum_{n=1}^{\infty}\frac{1}{2^n} \to 1$
Limit Value:	1



Earlier it was discussed that an infinite series can 'coverage' to a finite value which seems counterintuitive. The geometric series to the left gives a visual 'picture' of convergence: a unity box is divided it in half, then again and again… Each successive term has a common ratio of ½ - the tell tail sign of a geometric series (note: not all geometric series are convergent).

Series:	Reciprocals of Factorials
Type:	Convergent
Sequence:	$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$
Summation: Notation	$\sum_{n=0}^{\infty}\frac{x^n}{n!} \to e^x$
Limit Value:	e^x

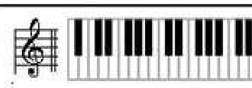


The constant e, base of the natural logarithm, can be expressed as an infinite series of the reciprocals of factorials. This is an important function to look at: a special infinite series that's not affected by differential and not by integration. The exclamation mark after the number means to multiply it by all the numbers below it (4! = 4x3x2x1). Without going into the grubby details, if we differential the above series, we get the original series (!). When x=1, the expression is 2.7182818… The number is called e. If we differentiate e^x we get e^x (the slope of e is e!).

e is used in phasor notation, probability theory, calculus, compound interest and more. In 2004, Google filed to go public with an unusual e billion-dollar auction offering ($2.718.. billion).

Harmonic Series

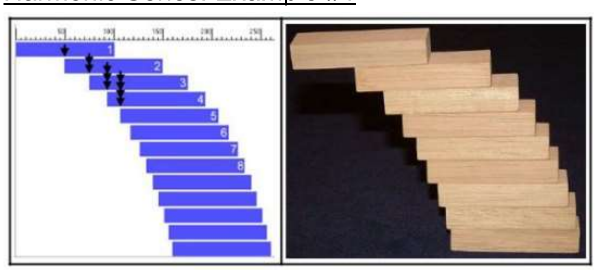
Series:	Harmonic
Type:	Divergent
Sequence:	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$
Summation: Notation	$\sum_{n=1}^{\infty}\frac{1}{n} \to \infty$
Limit Value:	∞



Key #	Note	Frequency (Hz)
13	A	55.0
25	A2	110.0
37	A3	220.0
49	A4	440.0
61	A5	880.0
73	A6	1760.0
85	A7	3520.0

The harmonic series derives from the concept of overtones, or harmonics in music. The wavelengths of the overtones of a vibrating string are 1/2, 1/3, 1/4, etc. of the string's fundamental wavelength. Every term of the series after the first is the harmonic mean of the neighboring terms.

Harmonic Series: Example #1



Example of applying the harmonic series to cantilever blocks. Given a collection of identical dominoes, using 'harmonic positioning', it is possible to stack them at the edge of a table so that they hang over the edge of the table without falling. The counterintuitive result is that one can stack them in such a way as to make the overhang arbitrarily large, provided there are enough dominoes.

[1] 2015 estimate was 19 million.

Harmonic Series: Example #2

Another problem involving the harmonic series is the ‘Jeep problem’, otherwise known as the desert crossing problem or exploration problem. The problem is to maximize the distance the Jeep can travel into the desert with a given quantity of fuel. The jeep can only carry a fixed and limited amount of fuel, but it can leave fuel and collect fuel at fuel dumps anywhere in the desert.

Determining Pi with Infinite Series

In 1985 pi was calculated to 17 million digits using the Ramanujan formula (developed in 1910). The Ramanujan formula was ideal for computers since pi converged exponentially compared to other algorithms. In 1989, the Chudnovsky brothers published the Chudnovsky algorithm (based on the Ramanujan formula). As of December 2013, the Chudnovsky algorithm holds the world record calculations of pi at 12.1 trillion digits.

Popular Series Formulas for Pi

Ramanujan	Chudnovsky
$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \times \frac{26390n + 1103}{396^{4n}}$	$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545140134k + 13591409)}{(3k)!(k!)^3 (640320)^{3k+3/2}}$

The Bailey–Borwein–Plouffe (BBP) formula (1995) lets you skip straight to any digit of pi without working out the rest of the number - a little bit of math sorcery.

Bailey–Borwein–Plouffe (BBP) formula

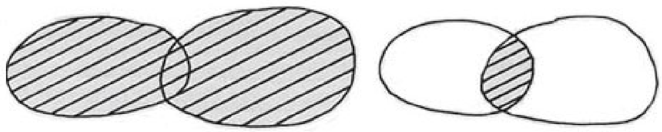
$$\pi = \sum_{k=0}^{\infty} \left[\frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right) \right]$$

Different Approaches to Algebra

- **Enclosures**

Venn Diagrams

Venn Diagrams



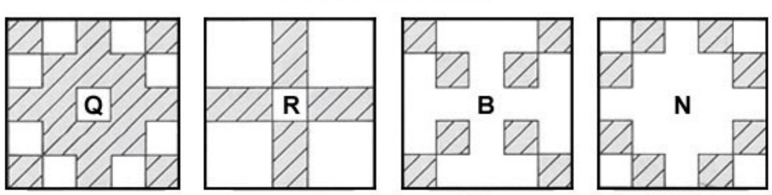
Union = A ∪ B **Intersection = A ∩ B**

Venn Rules

$a + b = b + a$	$A \cup B = B \cup A$
$a + (b + c) = (a + b) + c$	$A \cup (B \cup C) = (A \cup B) \cup C$
$ab = ba$	$A \cap B = B \cap A$
$a(bc) = (ab)c$	$A \cup (B \cap C) = (A \cap B) \cap C$
$a(b + c) = ab + ac$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

“Venn Chess” - Venn rules applied to chess.

Chess Piece Moves



Queen **Rook** **Bishop** **Knight**

Venn ‘rules’ for Chess

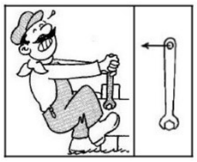
∪	Q	R	B	N
Q	Q	Q	Q	S
R	Q	R	Q	
B	Q	Q	B	
N	S			N

∩	Q	R	B	N
Q	Q	R	B	O
R	R	R	O	O
B	B	O	B	O
N	O	O	O	N

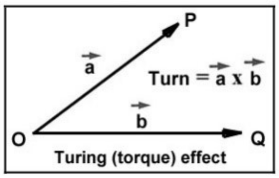
Are we still doing algebra? It is a matter of standpoint, but the mathematician would say we are. Continuing on with looking out of the ‘algebra’ box…

♦ **Vectors**

Things that must be described by more than one number are called vectors: a pair of Levis (length & waist), a bolt (length, diameter, threads/inch), a street intersection by two streets. Things that are described with only one number are known as scalars, like weight, or speed.



Let's consider a mechanical or engineering notion. When a force acts on something that is fixed at one point and acts off center - it tends to turn.

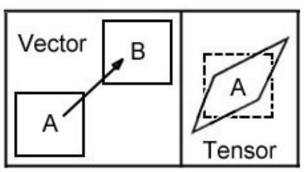


Turing (torque) effect

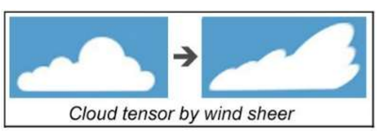
Mathematicians (and physicists) use an 'arrow' to describe a force. The formal name for the arrow is a vector. The direction of the arrow shows the direction of the force, and its length shows the strength of the force (velocity is also represented by a vector). The turning effect is expressed using vector mathematics (vector algebra) which have their own rules.

Since a vector represents a number *and* a direction they do not behave exactly like numbers - vectors do not have a commutative law for multiplication (ab ≠ ba). The first introduction of non-commutativity came from William Rowan Hamilton ('Hamiltonian Mechanics') and his development of quaternions, a number system that extends complex numbers into three-dimensional space (quaternions calculations are found in the 3D rotations of computer graphics, like in video games). It was after 15 years of intense intellectual effort that Hamilton considered the idea that an algebra could be consistent even with a non-commutative law.

Tensors

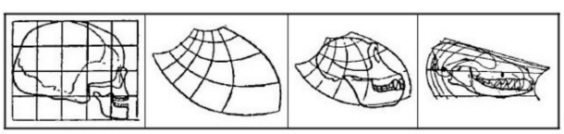


Moving a square's position from A to B can be represented by a vector. If square A is pulled into the shape of a parallelogram it needs to be represented by what is known as a tensor (the name comes from the tension that deformed the square). A tensor is made up of more than one vector. Using tensor 'algebra' a square would be represented by the tensor (↑,→) and the parallelogram by the tensor (↗, ↗). Each vector is represented by a string of scalar numbers, like (3, 5, 7).


Cloud tensor by wind sheer

If this 'tensor talk' sounds a bit out of this world, the next time you look up at the clouds, sometimes you see tensor transformations going on!

Super Tensors



Ordinary tensors can change the direction and length of straight lines but cannot *curve* straight lines. Super tensors can change straight lines into curved lines, as well as change their direction and length. Super tensor is a vector that is made of a string of ordinary tensors. The transformation of a human skull into the skull of a baboon or dog can is an example of a super tensor.

Super tensor equations were essential to Einstein's General Theory of Relativity which made the amazing connection between a geometric view of the universe with a stress-energy physical view of matter (that matter influences space and vice-versa!). Einstein's Field Equations (EFE) updated Newtown's description of gravity with the new vocabulary of 'spacetime' and the wild concept of spacetime curvature.

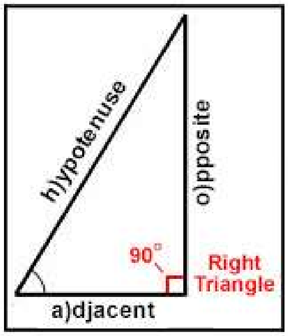
Quite often a mathematician has no notion that his algebras might fit in a real model and perhaps he did not care. With that said, our wily mathematician could get a little ‘crazy’ and imagine a 'super-super tensor' - vectors made of a string of super tensors!

Algebra’s Elements

The essential feature of mathematics is that the rules cannot produce a contradiction - they must be internally consistent. Sounds a little like philosophy, but that’s a topic for later.

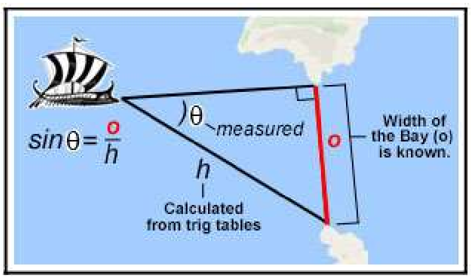
Algebra = Some Elements + Some Operations + Some Rules

Trigonometry



The term “trigonometry” is derived from the Greek word meaning "triangle measuring". In general, it is the study of a particular type of triangle: a three-side figure with one 90° angle – known as a right triangle.

The history of triangles dates back to antiquity - the right angle was an indispensable tool for Egyptian and Babylonian builders.



Our modern word "sine" is derived from the Latin word sinus, which means "bay". Early navigators, knowing with width of a bay, could determine their distance to a point using simple navigation tool and trig tables.

The word “tangent” comes from Latin tangens meaning "touching", since the line touches the circle of unit radius.

Sine Calculation Example

Width of Bay (o): 2,300 feet
Measured Angle: 35.4 deg
Sin(35.4): 0.5793 (trig table)
h: 3,970 feet

Trig Table

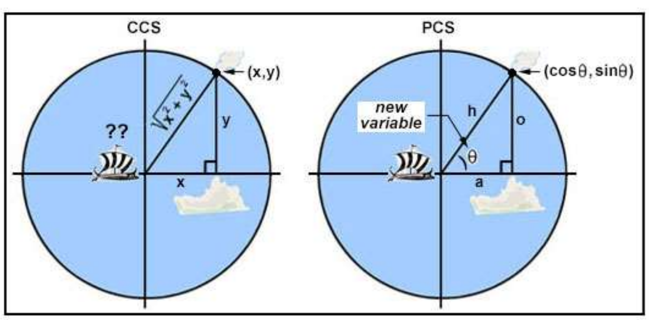
θ	SIN(θ)
35	0.5736
36	0.5878

35.4° = 0.5793 (interpolation)

h = 2,300ft/0.5793 = 3,970 feet

Note: the other angle is calculated by knowing that all the angles of a right angle add up to 180 degrees: 180° - 90° - 35.4° = 54.6°. Further, the adjacent distance is be calculated by using the Pythagorean Theorem ($h^2 = o^2 + a^2$). $a^2 = (3{,}970)^2 - (2{,}300)^2$. $a = (10{,}470{,}900)^{1/2}$ = 3,236 feet.

Cartesian Coordinate System (CCS) versus Polar Coordinate System (PCS)



The illustration to the left illustrates the benefit of using the Polar Coordinate System (PCS) over the Cartesian Coordinate System (CCS). Using the navigation example, the ship is at the center of a circle. One of the legs of a right triangle is the width of the bay. PCS introduces a new variable – the *angle*.
The “PCS” ship uses the angle (and trig table) to determine the length of the other leg(s) (distance from land).

The trigonometric functions are the ratios of two sides of a right triangle containing the angle. There are three main functions: Sine, Cosine and Tangent. The reciprocal functions are cosecant, secant and cotangent (the prefix "co-" refers to complementary angle).

Function	Ratio	Reciprocal Function	Ratio
sine	$\frac{\text{opposite}}{\text{hypotenuse}}$	cosecant	$\frac{\text{hypotenuse}}{\text{opposite}}$
cosine	$\frac{\text{adjacent}}{\text{hypotenuse}}$	secant	$\frac{\text{hypotenuse}}{\text{adjacent}}$
tangent	$\frac{\text{opposite}}{\text{adjacent}}$	cotangent	$\frac{\text{adjacent}}{\text{opposite}}$

<u>Formalization of the Trig Table</u>

The early astronomers created trig tables by using the sum-angle formula for sines: sin(A+B) = sinAcosB + sinBcosA. The process to obtain values was 'bootstrapped' from known values like sin(45°) or sin(30°). The sine of 72° was determined from Euclid's 5-pointed star. The sine of 3° was determined from the sines of 75° and 72° (Ptolemy figured this out in the 2nd century AD). The solutions, especially for degrees less than 3°, were time consuming and not very accurate or elegant.

Mādhava of Sangamagrāma (1340-1425), Indian mathematician and astronomer, was the first to use the infinite power series to determine the values of trigonometric functions (Power Series, pg. 18). A less time consuming and elegant solution.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Trigonometry reached its modern form from the mathematics of Leonhard Euler (1707-1783). His famous "Euler's Formula" correlated trigonometric functions to complex numbers: $e^{ix} = \cos(x) + i\sin(x)$. In the mathematical series section (pg. 19), e^x is expressed as an infinite series of the reciprocals of factorials.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Reciprocals of Factorials

$$\boxed{\begin{array}{ll} i^1 = i & i^3 = -i \\ i^2 = -1 & i^4 = 1 \end{array}}$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots$$

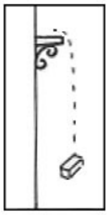
$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots$$

$$e^{ix} = \Big(\underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}_{\cos x}\Big) + i\Big(\underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}_{\sin x}\Big)$$

$$e^{ix} = \cos x + i \sin x$$

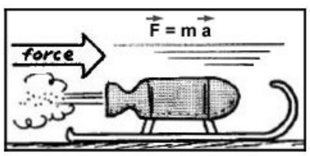
CH8 – THE CALCULUS

The word calculus comes from the Greek word for a pebble, or a stone used in calculation. There are some difficult ideas at the very beginning of the calculus, and if we get too deeply involved with infinitesimals we can be led out of our depth. Some of the notions of calculus are not easy to square with our basic intuitions. In short, calculus is used to study *how things change*, like the movement within space.



A stone falls. Aristotle held that how *fast* the stone fell depended on how *heavy* it was and that in fact it was proportional to its weight. That was not only untrue, it did not begin to discuss what was meant by ‘fast’. Science is about *prediction* and scientific prediction lies in *experiment*.
In studying the motion of objects Galileo wanted to know where the stone was at any time, how fast it was going and how it was accelerating – very advance questions that Aristotle and his contemporaries did not ponder.

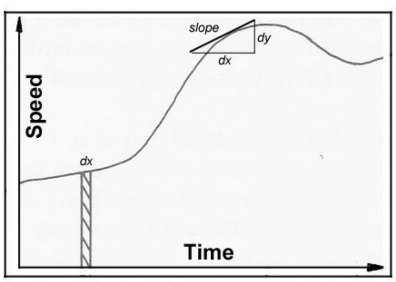
Sir Isaac Newton (1642-1727) remarked that if he had seen further than others it was because he had stood on the shoulders of giants, like Galileo. Newton’s skill lie in sorting through vast amount of other’s observations (ie, Galileo, Kepler) and reducing it, by asking ‘why’, to simple rules.



Newton said that forces were responsible for motion. That a constant force produces a constant acceleration. That was the surprise. Even today, ask an ordinary person what a constant force would produce and he will think it to be a constant speed. But we experience an acceleration when we put our foot on the accelerator and pull away in a car (thinking vs. experiencing).

The experience of seeing a stone skimming across a slippery icy pond is nearer the statement made by Newton that when there is no force then the object moves at steady speed (or stays still).

Motion under gravity is one of the few situations where the acceleration is steady.



If we look at the speed of driving over time it might like something like the curve to the left.

Differential Calculus
1) Acceleration: Slope of the Curve – dy/dx
Integral Calculus
2) Distance Traveled: Area under the Curve – $\int f(x)dx$

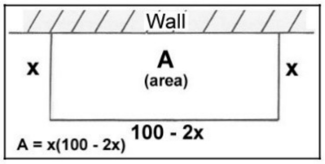
Differential calculus aims at calculating the slope – of finding the line called the tangent (Latin *tangere*, to touch). The process involves of taking smaller and smaller gaps around the point to find the slope which ultimately leads to the term ‘infinitesimal’ calculus (a rigorous proof of infinitesimal calculus is intricate and beyond this summary).

The operations of differentiation and integration is similar to remembering the multiplication tables. For example, the derivative of x^2 is $2x$, of x^5 is $5x^4$, of sin(x) is cos(x) and of cos(x) is –sin(x). That the integral of x^2 is $x^3/3$ (+ a constant) and of x^5 is $x^6/6$ (+ a constant) and so on. The important discovery is that integration and differentiation are inverse processes (like multiplication & division, to a degree). This fact is known as the fundamental theorem of the calculus.

Adding, multiplying, dividing and subtracting are operations on numbers. Integrations and differentiation are operations on expressions or functions.

Calculus Applications

Application #1



Situation: Plot of land and a wall.

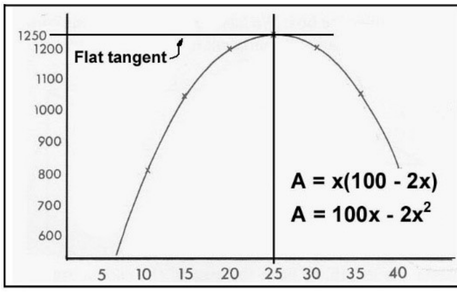
Given: 100 ft. of fence.

Problem: Enclose as much land with the fence in a rectangular shape against the wall. Calculus 'maxima' problem.

First, let's solve the problem with the 'Trail & Error' method: put in various values for x and find the maximum area. We find that when x=25, we get the maximum area (A=1250).

x	5	10	15	20	**25**	30	35	40
A	450	800	1050	1200	**1250**	1200	1050	800

To solve it formally using calculus, it's helpful to get a visual of the area equation. The principle depends on the picture, but the method is precise. What is more important is that it can be applied to examples where trial and error on graphical methods is not effective (see next problem).



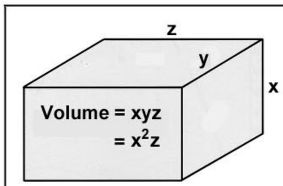
The area goes up to a peak and then comes down - smoothly & symetrically.
Calculus tackles this problem by looking at the peak and seeing that the tangent is flat - this is where the derivative (slope) is zero.

1) $A = 100x - 2x^2$
2) $dA = 100 - 4x$
3) Set derivative = 0
4) $100 - 4x = 0$
5) $x = 25$

Application #2

Situation: Postal regulations state that for a rectangular box the sum of the length and girth must not exceed 10 ft.

Problem: What is the largest volume you can send by post?



There's no reason why x or y should be bigger than the other; intuition tells us a square would be best (could be proved using the math of Example #1).

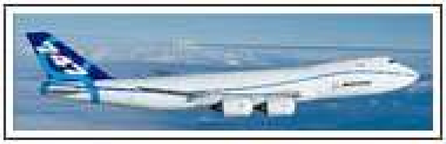
Box Dimensons

$\boxed{1\frac{2}{3}\text{ft} \times 1\frac{2}{3}\text{ft} \times 3\frac{1}{3}\text{ft}}$

1) Girth = $2(x + y)$
2) Let $y = x$ (square)
3) Girth = $4x$
4) $z = 10 - 4x$
5) Volume = $xyz = x^2(10 - 4x) = 10x^2 - 4x^3$
6) Derivative of Volume = $20x - 12x^2$
7) Set the derivative = 0
8) $20x - 12x^2 = 0 \rightarrow x = 5/3 = 1\frac{2}{3}$ ft

Other calculus examples?

Rocketry. The amount of fuel in a rocket determines how far it will travel (eg, parking orbit of the Apollo Saturn V missions). As the fuel burns the spacecraft gets lighter. With less mass it can travel farther. Unlike a car's gas tank, the total travel distance from a rocket tank of fuel is more complex – necessary to use calculus to determine the total amount fuel required when the changing mass of fuel affects the overall distance. Probably why they call it 'rocket science'.



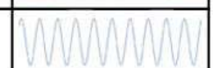
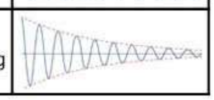
Jet Aircraft. The calculus of fuel burn-rate is found in the control of commercial jet aircraft. The maximum total range of the aircraft is limited by fuel capacity of the aircraft. The Breguet Range equation is summarized as the rate at which fuel is burned equals the rate at which aircraft weight is reduced: $dW/dt = -fgT$ (W = weight, T = Thrust, f = mass of fuel burn per unit of thrust per second). The calculus of the fuel burn/weight property is incorporated into the aircraft's 'auto cruise' system which controls the throttle to maintain the targeted air speed. Very helpful in reducing the pilot's work load.

Other down-to-earth examples of calculus include: cruise control system in your car; smart thermostat in your home, the graphics behind gaming software.

Two Very Important Differential Equations

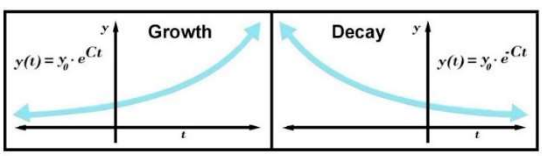
Calculus shows up behind two common and important behaviors in nature:
1) Natural Exponential Function.
2) Simple Harmonic Motion (SHM).

Starting with the general second-order differential equation $Ay'' + By' + Cy = 0$ a matrix is created with various constant values (A, B, C):

$Ay'' + By' + Cy = 0$							
Case	**A**	**B**	**C**	**Differential Eqn.**	**Solution Equation**	**Curve**	
①	**0**	**1**	**±1**	$y' = \pm Cy$	$y(t) = y_0 e^{\pm Ct}$ + = Growth − = Decay	Exponential	
②	**1**	**0**	**1**	$y'' = -Cy$	$y(t) = C_1 \cos(wt) + C_2 \sin(wt)$ $w = \sqrt{C}$ = natural freq.	SHM	
③	**1**	**1**	**1**	$y'' = -By' - Cy$	$y(t) = Z_1 e^{-t/2} \sin\left(\frac{\sqrt{3}\,t}{2}\right) + Z_2 e^{-t/2} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$ $Z = f(B,C)$	SHM with Damping	

$y' = dy/dt$ | $y'' = d^2y/dy^2$

Case #1: Natural Exponential Function

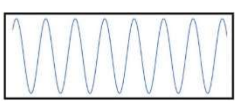


When the *first* derivative or rate of change of a quantity is proportional to *itself* the result is exponential. When the constant of proportionality is positive (+C) the quantity undergoes exponential growth and exponential decay when the constant is negative (-C).

Applications:

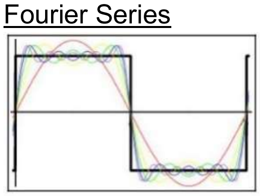
- Life science: growth of microorganisms, pharmacology & toxicology.
- Physical science: geophysics, heat transfer, luminescence, chemical & nuclear chain reactions, optics, thermoelectricity, vibrations (mechanical & electrical).
- Social science: finance (eg, interest rates).
- Computer science: Internet routing protocol.

Case #2: Simple Harmonic Motion (SHM)



When the *second* derivative of a quantity is proportional to *itself* times a negative constant, the result is an oscillatory motion that is sinusoidal in time known as Simple Harmonic Motion. SHM occurs when the force on a body is not constant (eg, elastic restoring force of a spring). In Case #3 the quantity is also influenced by the first derivative (y') which produces a dampening (or growth) effect. SHM examples in everyday life: pendulum, swing, car shock absorber, musical instruments, bungee jumping, hearing & rocking of the baby cradle.

Fourier Series



A very powerful mathematical tool is the Fourier series. In essence, the Fourier series decomposes a *periodic* function (or signal) into an infinite sum of sines and cosines (trigonometric polynomial).

The study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually (analysis), and then recombined to obtain the solution to the original problem (synthesis). Periodicty can be in time (eg, SHM) or in space (eg, heat distribution on a circular ring). Fourier analysis is often associated with symmetry.

Strawberry Smoothie	Ingredient Filter	Amount
→	Strawberry →	2 oz.
→	Banana →	1 oz.
→	Orange Juice →	6 oz.
→	Sweetener →	1 oz.
→	Protein Powder →	1 oz.

In layman terms, the Fourier Transform (FT) finds a recipe. For example, to determine the recipe of a smoothie, the FT reverse-engineers the recipe by filtering each ingredient. It's important that each filter be independent. The orange juice filter must capture only orange juice, nothing else. If the filters behave correctly, we can get back the original smoothie by blending the ingredients.

Fourier Series Equations

Trigonometric Form	Complex Exponential Form
$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$	$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$	$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega_0 t}dt$

The above equations look awfully complex, and they are to an extent. The point is to illustrate the connection between infinite series and calculus.

<u>Taylor Series</u>

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Taylor Series

Another powerful tool that's analogous to the Fourier series is the Taylor series which represents a function as an infinite sum of powers rather than an infinite sum of sines and cosines. The coefficients are calculated from the function's *derivatives*, $f^{(n)}$, at a single point a.

$$\boxed{\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots}$$

Power Series

The Power Series (pg. 18) is the 'reference' equation for an infinite polynomial function. A convergent power series is known as an 'analytic' function which is infinitely differentiable. A mathematician would say, "a function is analytic if and only if its Taylor series about x_0 converges to the function in some neighborhood for every x_0 in its domain (the set of x).

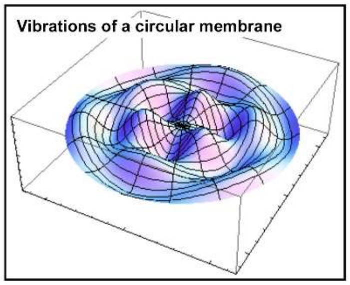
Elementary Analytic Functions: 1) all polynomials, 2) exponential function, 3) trigonometric functions, 4) logarithm functions, 5) power functions

Special analytic functions (at least some in some range of the complex plane):
1) Hypergeometric Functions
Determines the solutions of a second-order linear ordinary (single-variable) differential equation (ODE):

$$\boxed{\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)}$$

2) Bessel Functions
Determines the solutions of Bessel's inhomogeneous second-order differential equation:

$$\boxed{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0}$$



Bessel functions are useful in solving problems of wave propagation: electromagnetic waves in a cylindrical waveguide, pressure amplitudes of rotational flows, heat conduction in a cylindrical object, modes of vibration of an acoustic membrane (eg, drumhead, figure to the left).
The Bessel function also provides solutions to the radial Schrödinger equation for a free particle (definitely 'Grad' level math).

3) gamma function
Factorial of non-integers and complex numbers. Used in probability and statistical models.

CH9 – TAKING A CHANCE: LIVING WITH UNCERTAINTY

"It would be very unexpected for the unexpected not to happen sometimes". – Isaac Asimov

A scientific study of uncertainty involves the mathematics of probability. When an insurance company takes a bet with you as to how long you are going to live, it does not know whether it will win or lose (it loses if you die early – you win, kind of). Nor is it concerned whether it wins or loses with you. The number of people with whom it has similar bets makes the insurance company sure of its eventual financial gain. This might sound macabre, talking about betting on one's life, but that is what life insurance is. And the insurance companies have spent a lot of time and effort on actuarial work – the measurement and management of risk and uncertainty.

In analyzing trends, economic or political, we cannot say what will happen, but we can assign probabilities on the possible outcomes.

The rule-of-thumb principle called the Pareto Principle states that, for many events, roughly 80% of the effects come from 20% of the causes – "80-20" rule. The Pareto Principle can be a very effective problem-solving tool. For example, where to focus the first efforts of an investigation.

Pareto Principle Examples:
1) 80% of the land is owned by 20% of the population
2) 80% of the peas come from 20% of the peapods.
3) 80% of a company's profits comes from 20% of its products.
4) 80% of complaints come from 20% of the population.

Karl Popper (philosopher of science, 1902-1994) referred to social philosophies that possessed predictive abilities as *historicism* - outside the realm of as empiricism and rationalism. Historicism neglected the role of traditions (eg, the interior of the collective). His main objection was against the assertion by some that we are at the mercy of trends and cannot resist them. However, Popper attempted to make a clear distinction between scientific prediction and historical prophecy.

The trauma of wanting to be sure that what we are doing is right is not one that the probabilistic gets involved with. He or she knows that certainty is not possible. You take the 60% route and do not fuss. That is the right and sensible thing to do. Sometimes the "40%" comes up as it must do 40% of the time.

Or that the odds were in fact 40-60, not as we thought. That's depends on the information available to us. The 'take home' message is only to have regrets if we have failed to take into account things that we could have known before the decision, or that we wrongly judged what we did know. It takes a good deal of internal security to live with uncertainty, but it can be quite comforting when achieved.

CH10 – MODERN MATHEMATICS

"Thus mathematics may be defined as the subject where we do not know what we are talking about, neither do we know if what we are saying is true." – Bertrand Russel

Sets

It is the common perception that the value of mathematics in the everyday world lies in calculation. It is useful, but so is sorting.

"Sets' is one trigger word for Modern Mathematics. Examples of significate sets:

1. Everyday sorting – shapes to warehouses to libraries.
2. The beginnings of any science – the material is sorted into patters to find relevance.
3. Objective knowledge and the philosophical bias of number[2].
4. Early concept formation and the beginnings of language.
5. Comparisons of cognitive power.
6. The logic that resides in particular words within language.
7. Computer language.

Russell and Whitehead started with undefined sets of objects and used the idea of pairing off which established a one-to-one correspondence between the objects of the two sets.

It is now common practice in infants' schools to approach number in this way. Children are encouraged to collect sets of five object (paintbrushes, pencils, chairs, friends) to establish the "fiveness of five".

The process of making sense of our environment relies upon sorting and ordering what we see, hear, smell, taste and touch. An essential part of this process for the human being is the language that goes with this sorting and ordering. There seems to be an intimate relationship between early concept development, sorting (mathematical issue) and naming (the start of language). There is ground for supposing that in our psychological processes, early mathematics, and language are closely knit.

Sets have a relation to 'intelligence'. "What it is" and "how it develops" is a topic fraught with rational difficulties and bedeviled by emotional responses. The ability to sort is clearly different in different individuals. The ability to perceive samenesses and differences seems to be one aspect of cognitive power.

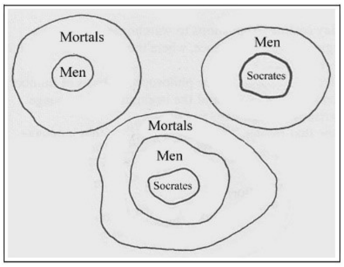
George Boole published "The Laws of Thought" (1854) - established Boolean algebra.

Mathematics abstracts the relationships that exist. It removes the issue of what we are talking about, but is concerned with one idea lying within another, overlapping it, or being entirely distinct, as in a pictorial diagram. We are now better able to understand Russell's statement. We do not know what we are talking about, for the nature of the sets is not the issue. It is a matter of how they relate one to another. Nor are we concerned with truth.

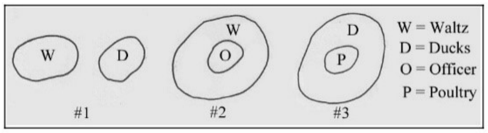
The theme of mathematical logic runs through the stories of Lewis Carrol (Alice in Wonderland), and some of Carrol's syllogisms were quite complex.
Syllogisms, dating back to the Grecian philosopher Aristotle, expressed what logic is about. In regard to proof, Aristotle's 'logic-chopping' methods of syllogism is best understood as achieving a 'bare proof'. When Aristotle says, 'Proof is reasoning that causes us to know' is to be understood, in the modern sense, that the proof is accompanied by appropriate experience.
Classical syllogism logic:

Statement	Logic Structure	Observation
All men (A) are moral (B).	A = B	The General.
Socrates (C) is a man (A).	C = A	The Particular.
Socrates (C) is mortal (B).	C = B	The drawn conclusion

[2] "God made the integers; the rest is the work of Man." – Knonecker.



Venn diagrams can be a useful sorting mechanism. The diagram at the left shows Aristotle's syllogism explained with a Venn diagram:



Lewis Carrol offered as many as ten statements from which to draw a conclusion. Here's an example of three statements:
1) Duck do not waltz. 2) No officer ever declines to waltz.
3) My poultry are all ducks.

The conclusion 'My poultry are not officers' can be visualized with the Venn diagram:

III – WORKING WITH MATHEMATICS
CH11 – MATHEMATICS IN ACTION

Pure vs. Applied Math

At one time a sharp distinction was made between 'pure' mathematics and 'applied' mathematics. One can start in pure mathematics (conducted for its own sake) and ended up in some issues that are totally related to this world and the universe in which it moves. Einstein said: "Is it not remarkable that mathematics, a study independent of experience, should be so adapted to the object of reality?" There is something mysterious about it. No pure mathematician can ever guarantee that that his work will not be of practical use.

The range of algebras developed by Grasseman and his development of Tensor Calculus later gave a deeper insight of space and time – a symbol system with strange rules that could in no way be expected to relate to anything in the physical world.

Einstein's mathematics of relativity and the later discovery that his predictions of Mercury's path at its perihelion were closer to the truth than the mathematics of Newton.

The reality is that mathematics, and science for that matter, answer only a small fraction of all the possible questions we can ask (Godel's theorem set limits on how much we can actually prove). Mathematics can appear to have the illusion of success if we are preselecting the subset of problems for which we have found a way to apply mathematics. One example is the impressive progress with linear systems. On the other hand, developments in nonlinear systems have been arduous and much less successful. If we focus our attention on linear systems, then we have preselected the subset of problems where mathematics is highly successful.

The reader is now asked to entertain a strong non-Platonism position where all physical laws are tainted with anthropocentrism and all physical models have no real interpretative value. The interpretive value of physics is purely illusory. After all, a beam of light passing through a slit knows nothing of Fourier transforms. This is an overlay of human construct.

Mathematical Modeling

John von Neumann, the famous mathematician and polymath, stated all this more succinctly: "The sciences do not try to explain, they hardly even try to interpret, and they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justifications of such a mathematical construct are solely and precisely that it is expected to work."

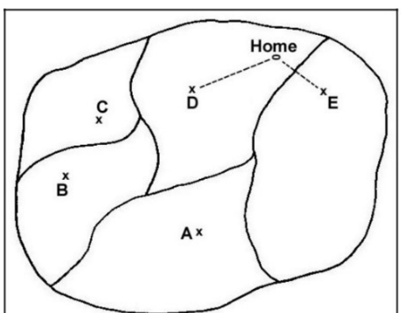
With the success of the mathematical models in physics it's easy to overlook where they don't work well. Like in weather forecasting. Predictions are typically good for a week. For longer forecasts, small errors grow into big ones. Daily weather is just too complex and chaotic for precise modeling. So is the behavior of water boiling on the stove or the stock market or the interactions of neurons in the brain, much of human psychology and parts of biology. Biological, environmental and economic systems are very difficult to model with mathematics. The real world is inherently noisy and has a stochastic component (randomness) so the math can reach the height of intractability: stochastic, non-linear, partial differential equations with non-linear coefficients!

When classifying physical data, it is known that 'God does not always shave with Occam's razor'.
(H. Bensusan article, "God doesn't always shave with Occam's razor – Learning when and how to prune").

Discovering & Describing Patterns
Mathematics is a human invention for describing patterns and regularities. It follows that mathematics is then a useful tool in describing regularities we see in the universe.

A skill of great importance in mathematics, and in many other studies, lies in seeing that problems are of the same type, that they have the same 'shape'. The term used in mathematics is 'isomorphic', from Greek meaning 'having the same shape'. One new branch of isomorphic mathematics is known as Operational Research (OR) which attempts to maximize performance in certain systems under given constraints. Examples includes from locating a lost submarine to finding a lost ring. Mathematician J.D. Bernal is recognized as OR's early pioneer and significate contributor.

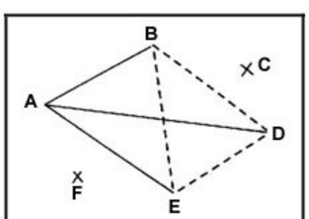
The mathematician is sometimes accused of treating people as numbers or as things. The individual mathematician may or may not do so. It's just the objective nature of properly solving a problem, like queuing theory - queues waiting to get into a hospital or the queues on the production line. The worst situations are often achieved by muddled sentimentalists who are seized with the problem of one individual and allow them to jump the queue thereby causing more problems to others. Unrestrained sympathy, unrelieved by any rational process, is not the best way to run an organization which needs to be both sympathetic and fair.



School Catchment Area

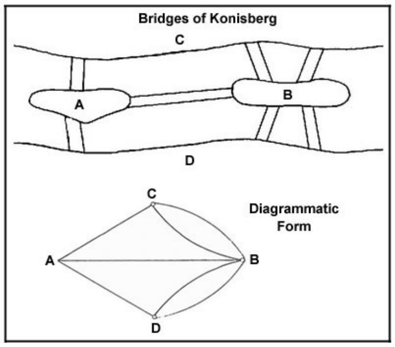
Another category of problem is concerned with allocation, best illustrated with a school system 'catchment' area. In an urban area large enough to support five secondary schools, catchment areas are show to the left. From the standpoint of the school district authorities, the catchment area boundaries reflect constraint logistics such as roadways and school size capacity. For some parents catchment anomalies challenge the school's logic, as seen in catchment area D where the home is closer to the school in area E. If the parents strongly desire school E (with good reason) the dictate of the school authority does not make sense.

One of newest branches of mathematics is Graph Theory (not the type of graphs seen in algebra). Graph Theory falls in domain of 'discrete' mathematics. Basically, it is the study of points jointed by lines.

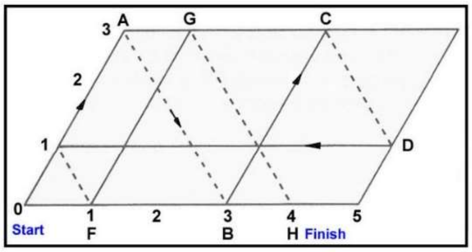


Friends & Strangers

Example 1 Six people gather together. Prove that there are either three who all know one another, or three who are all strangers.



Example 2 Bridges of Konigsberg. The river flowing through Konigsberg has two islands in it, the islands being connected to each other and the banks of the river by a total of seven bridges. Is it possible to cross every bridge once and only once?

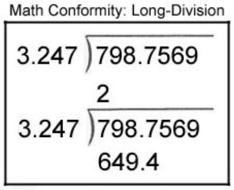


Example 3: Jugs of Wine Two men have a full 8-gallon of wine, and each man has an empty 5-gallon and 3-gallon jug respectively. What is the simplest way to divide the wine equally?
Solving the problem with graph theory uses an 'isometric' paper with equilateral triangles. The 'vertical' axis equals 3 and the horizontal axis equals 5. Start at (0,0) and travel to point 'A' to fill the 3-gallon jug. The solution is to find the best path to get to 4 gallons (A to H). Dash line = 3-gallon jug to the 5-gallon jug. Solid line = adding/subtracting from the 8-gallon jug.

CH12 – ALGORITHMS, PROBLEMS & PURPOSE

The word 'algorithm' derives from 'Al Kuwarizmi', which means 'the man from Kuwarizmi' – the surname of the Arab mathematician Abu Ja'far Muhammad ibn Musa. His mathematical work introduced the Arabic system of numeration to Europe and the word 'algorithm' became attached to the arithmetic process.
In modern times the word algorithm generally refers to any routine process by which you proceed from a question to an answer. And 'routine' is not necessarily restricted to arithmetic process; it might lie in any area of mathematics or might even be the set of rules by which you work out by rota (eg, hospital or railway rota). A computer cannot think, so it requires from you make a set of rules, appropriate to its inner properties. Before the affordable pocket calculator, the number 7987.569 divided by 32.47 was done by the algorithm known as long division. It was a tedious task that put people in the role of a machine.

Math Conformity: Long-Division



There is little virtue in teaching long division. It instructed the dull and repelled the intelligent. It engaged the conformer in a discipline in which if he continues to conform, he will fail, and it obscured from the creative the pleasure that they might derive from mathematics. In extreme cases, the presentation of mathematics as being based upon arbitrary rules, whose reason may not be sought, can be psychologically damaging.

There are ways of going about problem-solving. What makes it a problem is that there seems to be no special way of going about it. Here are some useful rules
:
1) Don't Start. No attempt should initially be made to reach a solution. Familiarized yourself with the problem and the problem statement.
2) Stabilize the Problem. Examine the problem from all angles. For example if the problem stated 'for every prime number', consider temporarily eliminating the word 'every'.
3) Try an Easy Case.
4) Reflect. Watch out for being too clever; review the initial premise; etc.
5) Draw a Diagram. Pictures can help (can't hurt).
6) Generalized. Focus on the general and diminish the specific.
7) Search for Patterns.
8) Consider the Complement.
9) Does it Remind You of Anything?.

As with a piano, a note struck elsewhere in the room sets a string vibrating in sympathy, so it is with problems that need divergent rather than convergent thinking. This leads us to what Edward de Bono calls ‘lateral thinking’.

Those who are ‘too clever by half’ (TCBH) do tend to solve many problems quickly. It does not follow, because they are swift, that they are necessarily deep thinkers. It may be this type of problem at which they excel, and the reason for the speed lies behind the slogan. In ‘Intelligence, Learning and Action’, professor Richard Skemp refers to the ability of ‘resonance’ – sudden insights that yield economical solutions. Insights that are influenced from one’s wide experience of problems and drawing of analogies between them.

Attempts to measure intelligence are doomed to failure. Some will find number problems easy and those involving visualization difficult; other will find the reverse. In regard to the general capacity of the two hemispheres of our brains, one side is believed to control linguistic and numerical abilities, where the other side involves spatial perceptions. In any individual the imbalance may be very marked.

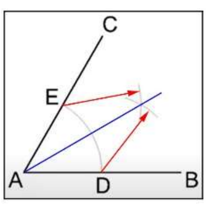
Mathematics is seen by many as made up of rules and formulae which constrict and constrain. This is because their main experience has been with algorithmic processes. Mathematics, properly understood, is an area where questions and issues needing answers constantly arise.

Examples:
1) Can we predict the number of open regions with any given number of lines?
2) Derive a formula for the number of different configurations in an n-line diagram.

There used to be a fair degree of effort in solving tedious, ‘algorithmic-like’ problems. With the affordable microcomputer, the donkey work disappears, and the amusement remains (see CH6, Fractals).

Resolution vs. Solution

History has shown us that many problems easily stated took centuries of effort to be solved. Had their impossibility been proved, that in itself would be a solution. A famous problem illustrates this point.



The Greeks found many geometrical constructions such as developing a method for bisecting an angle using only a straight edge and pair of compasses.

It’s been over a century now that the impossibility of trisecting an angle into three equal parts was established.

It came through algebra, not geometry, that if the problem turned out to be a quadratic (involving x^2) then it could be done with straight edge and compass. If of higher order, it could not. Trisecting an angle is equivalent to solving a cubic equation (involving x^3) and it cannot be done in the terms stated. Thus, mathematicians arrived at a ‘resolution’ of the problem, not a ‘solution’.

Another way to look at the problem-solving process:

1) Absolve. Ignore the problem (or delegate it). Many ‘problems’ reported, especially in the technical realm, are not real problems and quickly go away or the level of the reported severity (and consequence) is greatly reduced.

2) Resolve. Satisfy. Clinical approach: experiment, trial & errors; common sense; qualitative judgement. ‘Quick-fix’, ‘Band-Aid’ fix. Identify cause of problem and remove or suppress. Return to previous state.
Example: a product is not meeting is technical specification but represents only a meager fraction of the company’s business. Resolve: modify the spec sheet (judgement – the investment to correct the situation greatly exceeds the ROI).

3) Solve. Optimize. Research approach: formal experimentation, quantitative and formal (mathematical) analysis. Solution is the best of possible outcomes. What they try to teach in college.

4) Dissolve. Idealize. Redesign approach: eliminate the problem by approximated an ideal system. Do better in the future than the best that can be done now. Example: rather than widening a road to reduce traffic, the city government installs bike paths.

The Ineffectiveness of Mathematics

In 1956 mathematician Eugene Wigner, Princeton University, wrote a paper titled: "The Unreasonable Effectiveness of Mathematics in the Natural Sciences". Wigner's position was that the fact that mathematics can describe the universe so well in particular physical laws, is a gift that we neither understand nor deserve.

Particle physicists will say that particles are discovered mathematically. When a noteworthy particle physicists was asked, "How does this work? Is math a truth of nature or a way human's perceived nature?" His reply, "Fascinating puzzle, don't know the answer".

Not everyone agrees with Wigner's position. The famous mathematician Stephan Wolfram comments: "I think it is an illusion. What's happening is that people have chosen to build physics, for example, using the mathematics that has been practiced, and has been developed historically. They're looking at everything, they're choosing to study things which are amendable to study using the mathematics that happen to arisen. But actually, there's a whole vast ocean of other things that are really quite inaccessible to these methods."

Other responses have been that Wigner's idea that mathematics is a "miracle" is to suggest that the effectiveness is overstated. For example, the analytical equations that once described the physical properties of transistor behavior is no longer valid given the deep sub-micrometer dimensions of today's designs. The physics is just swamped with too many complicated higher order effects that can no longer be neglected at the small scale. Empirical models are now used in today's computer simulation software for circuit design. Traditional analytical mathematics simply fails to describe the system in compact form.

Another example is the use of Maxwell's equations for modeling integrated electromagnetic devices (ie, cell phones) and structures. In modern devices, due to the complexity of design, it is no longer effective to use analytical calculations. The standard approach today is the use of electromagnetic simulations programs that use numerical methods. The upshot is that when analytical methods become too complex, the pragmatic solution is to use empirical models and simulations.

IV – SUMMING UP

CH 13 – MATHEMATICS: ITS NATURE & PURPOSE

Society has always decreed that those who undertake education should have a substantial input of mathematics.

Teachers accept its importance, yet many of those who teach the subject would struggle to give a convincing rationale for what they teach. When the pupils ask, 'What use is this?' the answers provided are not always compelling. Yet our educational institutions state, always implicitly, and often explicitly, that the subjects that really matter are mathematics and our English language.

Convincing rationale example: mathematical modeling to predict electrical energy savings from Energy Conservation Opportunities (ECOs). Reduce carbon footprint (CH7, pg. 18).

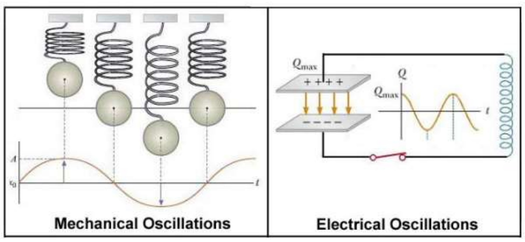
A story by the British playwright and novelist, W. Somerset Maugham: A churchwarden is sacked by a new vicar because he cannot read. The churchwarden had some savings, invested then in a shop, gradually built a chain of shops and became wealthy. On learning that he cannot read, the bank manager asks where he might have reached had he been able to. The man replies that he would be earning a pittance as a churchwarden. Moral: while he was not literate, he may well have been numerate.

We are all wonderfully flexible in the face of many sorts of disability, yet there are some mathematical needs that it is very difficult to do without. They are fewer than we might believe: to tell the time, monetary calculations and daily routines that involve sorting processes.

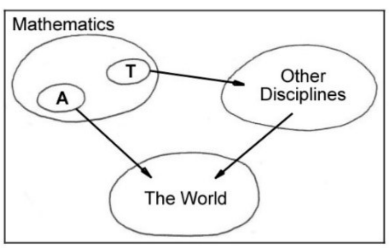
Direct Applications of Mathematics
1) Applied mathematics (Newton creating the calculus; part of the natural sciences).
2) Operational Research (matters involving process and how things are done; part of the social sciences).
3) Statistics
4) Topology
5) Graph Theory

Mathematics as a Language and a Tool
"Mathematics is the gate and key of sciences". Roger Bacon

A physicist nowadays cannot pursue his or her studies without extensive mathematical equipment. The concepts needed to understand relationships in both the hard physical world of concrete objects, and the web of forces and fields that seem to predict and explain what happens, are all expressed in terms of mathematical symbols and equations. The math has gone so far that there is an uneasy feeling that the mathematical symbols *are* the entities which physics discusses (since physics has become highly dependent upon mathematics).



Baconian Example: the mathematics of simple harmonic motion in both mechanics (eg, springs & weights) and electricity (charge flow Q in electrical circuits with capacitive and inductive components – the fundamental oscillators used in analog radios).



Conjecture: the nexus of mankind's intellectual progress will be between mind, language and mathematics.

Mathematics can be seen as a tool or a means of communication.
(A) = Mathematics applied to the world.
(T) = Mathematics used as a tool in other disciplines.

Mathematics for its Own Sake
Like linguistics, mathematics is capable of being highly narcissistic. Both are capable to work entirely inside their realm, oblivious of the outside world and make great discoveries within their boundaries. Once a 'language' has been created it is possible to work within it, not necessarily even being concerned with meaning.

Some creative mathematicians have been motivated by a desire to solve problems deriving from the real world; others have no interest in it – the mathematics itself suffices. A very pure mathematician would be content to say:

"Consider a set of elements a, b, c … and two operations * and /. They conform to the following rules…"

If you ask what these elements a, b, c… are, he will reply, 'they are defined by the connections enshrined in the rules'. If you say, 'What exactly are the operations? He will reply, 'The rules indicate what the operation are in terms of the elements.' In other words, elements and operations define each other by the rules governing them. The system is totally self-referential. It seems we may be in a 'black hole'.

Not quite true. The Greek's conic sections (CH5) and Grasseman's algebras were developed as a pure mathematical exercise and later found uses in the real world, thereby escaping from within mathematics.

Mathematics for Personal Development

Thinking is a process, not an appreciation of structure. Modern linguistic approaches to language do see it as a process as well as a structure. A mathematical example of a process is illustrated in Euclid's proof that the number of primes is infinite. If they stop, then consequence was that there was a higher prime and we got into a contradiction. Hence, they went on forever.

The processes of mathematics and of logic are the processes of the mind. In learning mathematics, we are matching our minds with the external manifestation of minds more powerful than our own. It is not exactly that mathematics trains the mind. Its processes are those of the mind. In the words of Lord Kelvin, "Do not imagine that mathematics is hard and crabbed and repulsive to common sense. It is merely the etherealization of common sense."

Caution: In justifying mathematics, overkill is easy, since no one is resisting.

Some Central Uncertainties

There are two dominate views of mathematics:

1) Platonic – see mathematics as external and to be discovered.

2) Formalist – regards mathematics as some form of game of chess with elements with relationships between them but no dependence the outside world.

The Platonic view saw geometry as the central truth, and it was not only true, but it also matched the space of the real world. That view was destroyed by the development of non-Euclidean geometries and the fact that some seemed to fit the world better than Euclid. This did not mean that it lacked an existence and a truth of a God-given sort, but the point-of-view simply became more abstract.

The formalist view (non-Platonic) will not accommodate the notion that intelligence cannot fail to develop number. We are not going to resolve these deep issues at the end of our last chapter but can state a position. Mathematics does have the objective nature, but that does not mean it depends in any way on the external world, nor does its testing reside there. That is internal. We have returned yet again to Einstein's question: *"How can it be that mathematics, a creation of the human mind independent of existence, should be so adapted to the objects of reality?"*

The Barber Paradox

In "Principia Mathematica" (1910) Alfred North Whitehead and Bertrand Russell establish that 1 + 1 = 2 (in 52 steps). In the analysis of sets, Russell became unsure of his notion of a set and devised statements with internal self-contradictions. A classic example is the "Barber Paradox": the barber is the "one who shaves all those, and those only, who do not shave themselves." The question is, does the barber shave himself?

Answering this question results in a contradiction. The barber cannot shave himself as he only shaves those who do not shave themselves. If he shaves himself, he ceases to be a barber. If the barber does not shave himself then he fits into the group (a set) of people who would be shaved by the barber (and, so, as the barber he needs to shave himself).

One way to get out of the paradox (logical 'jail') is to see that the Barber paradox is not really a paradox in the true sense of the word. A man who shaves *exactly* those men who do not shave themselves simply cannot exist, and there are no reasons to expect the opposite. This invalidates Russell's initial position: that the set of all sets do not contain themselves (avoids circular logic).

A final blow to certainties was struck by Kurt Gödel in 1931 with the publications of his incompleteness theorem on mathematical logic which basically shows that it is impossible to find a complete and consistent set of axioms for all mathematics (a la Hilbert's program).

Since Euclid mathematics had been concerned with axiomatic systems. One started with certain assumptions and built from there. For Euclid, there were 'self-evident truths' external to oneself. Whether a Platonist and a Formalist, in either view it was expected that you could build a structure such that if there were a proposition it could be proved or disproved.

Gödel proved that in an axiomatic system there would be propositions that were 'formally undecidable'. The proof is profound and perhaps one of the most shattering statements ever made in philosophy. Even

in relatively narrow areas of knowledge where the starting points and the rules of the game are clear. there will be statements whose truth or falsity (within the system) cannot be established. In other words, there are statements that cannot be proved or disproved. For fun, Google Gödel's statement, "True, but not provable".

"Maturity is the capacity to endure uncertainties" – John Huston Finley
The hope of most people is to have something to cling on to. Faith can remain, but it would be more sense if we had a basis in reason. The answer has to be in accepting *what is* and not longing for what we hoped might or ought to be. There are central uncertainties, yet it is possible to live with them, and still enjoy life and mathematics.

In Closing

In closing the last chapter of this review of mathematics, it cannot be hoped that we now know what mathematics is, but we must be content that we perhaps know more about it. We end with a list of quotations from Morris Kline's "Mathematics in Western Culture".

If, at the end, it is Huck Finn's view that most commends itself to you…so be it.

"In every department of physical science there is only so much science, properly so-called, as there is mathematics." – Emanual Kant

"Maths is the gate and the key of sciences…Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot view the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy." – Roger Bacon

"Do not imagine that mathematics is hard and crabbed and repulsive to common sense. It is merely the etherealization of common sense." – Kelvin

"Music is the pleasure the human soul experiences from counting without being aware that it is counting." – Gottfried Wilhelm Leibniz

"The science of Pure Maths, in its modern developments, may claim to be the most original creation of the human spirit." – A. N. Whitehead

"Geometry will show the soul towards truth and create the spirit of philosophy." – Plato

"Mighty is geometry; joined with art, resistless." – Euripides

"But where our senses fail us reason must step in." – Galileo

"I have never been able fully to understand why some combinations of tones are more pleasing than others, or why certain combinations not only fail to please but are even highly offensive." – Galileo

"For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity nor accommodated into use with sufficient dexterity without the aid and intervention of mathematics." – Francis Bacon

"How can it be that mathematics, a product of human thought independent of experience, is so admirably adapted to the objects of reality?" – Albert Einstein

"I had been to school…and could say the multiplication table up to 6 x 7 = 35 and I don't reckon I could ever get any further than that if I was to live forever. I don't take no stock in mathematics, anyway." – Huck Finn

"Besides the mathematical arts there is no infallible knowledge, except that it be borrowed from them." – Robert Recorde

"Nor should it be considered rash not to be satisfied with those opinions which have become common. No one should be scorned in physical disputes for not holding to the opinions which happen to please other people best." – Galileo

"In order to seek truth it is necessary one in the course of our life to doubt as far as possible all things." – Descartes

"All the pictures which science now draw of nature and which alone seem capable of according with observational fact are mathematical pictures." – James Jeans

"People who don't count won't count." – Anatole France

"Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." – Bertrand Russell